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Some cardinal and topological properties of the *n*-permutation degree of a topological spaces and locally τ-density of hyperspaces

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SOME CARDINAL AND TOPOLOGICAL PROPERTIES OF THE n-PERMUTATION DEGREE OF A TOPOLOGICAL SPACES AND LOCALLY τ-DENSITY OF HYPERSPACES

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Abstract

In the paper we investigate some cardinal and topological properties of the n-permutation degree of a topological spaces and locally τ-density of hyperspaces. It is proved that the functors $\exp_n$ and SP^n preserves locally τ-density of any infinite T_1-spaces.

Keywords: k-spaces, locally τ-density, hyperspace, n-permutation degree.
Mathematics Subject Classification (2010): 54-XX

1 Introduction

A permutation group X is the group of all permutations (i.e. one-one and onto mappings $X \to X$). A permutation group of a set X is usually denoted by $S(X)$. If $X = \{1, 2, \ldots, n\}$, $S(X)$ is denoted by S_n, as well.

Let X^n be the n-th power of a compact X. The permutation group S_n of all permutations, acts on the n-th power X^n as permutation of coordinates. The set of all orbits of this action with quotient topology we denote by SP^nX. Thus, points of the space SP^nX are finite subsets (equivalence classes) of the product X^n. Thus two points $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are considered to be equivalent if there is a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$. The space SP^nX is called the n-permutation degree of a spaces X [1]. Equivalence relations by which we obtained spaces SP^nX and $\exp_n X$, is called the symmetric and hypersymmetric equivalence relations, respectively. Any symmetrically equivalent points X^n are hypersymmetrically equivalent. But inverse is not correct. So, for $x \neq y$ points $(x, x, y), (x, y, y) \in X^3$ are hypersimmetrically equivalent, but not symmetrically equivalent.

The concept of a permutation degree has generalizations. Let G be any subgroup of the group S_n. Then it also acts on X^n as group of permutations of coordinates. Consequently, it generates a G-symmetric equivalence relation on X^n. The quotient space of the product X^n under the G-symmetric equivalence relation, is called G-permutation degree of the space X and is denoted by $SP^n_G X$. An operation SP^n_G is also the covariant functor in the category of compacts and is said to be a functor of G-permutation degree. If $G = S_n$ then $SP^n_G = SP^n$. If the group G consists only of unique element then $SP^n_G X = X^n$. Moreover, if $G_1 \subset G_2$ for subgroups G_1, G_2 of the permutation group S_n then we get a sequence of the factorization of functors:

$$X^n \to SP^n_{G_1} \to SP^n_{G_2} \to SP^n \to \exp_n.$$

Let X be a T_1-space. The collection of all nonempty closed subsets of X we denote by $\exp X$. The family B of all sets in the form $O\langle U_1, \ldots, U_n\rangle = \left\{F : F \in \exp X, F \subset \bigcup\limits_{i=1}^{n} U_i, F \cap U_i \neq \varnothing, i = 1, 2, \ldots, n\right\}$, where $U_1, \ldots, U_n$ is a sequence of open sets of X, generates the topology on the set $\exp X$. This topology is called the Vietoris topology. The $\exp X$ with the Vietoris topology is called the exponential space or the hyperspace of X [1]. Let X be a T_1-space. Denote by $\exp_n X$ the set of all closed subsets of X cardinality of that is not greater than the cardinal number n, i.e. $\exp_n X = \{F \in \exp X : |F| \leq n\}$.

Let's put

$$\exp_\omega X = \bigcup\{\exp_n X : n = 1, 2, \ldots\}, \exp_c X = \{F \in \exp X : F \text{ is compact in } X\}.$$

It is clear, that $\exp_n X \subset \exp_\omega X \subset \exp_c X \subset \exp X$ for any topological space X.

Definition 1 ([2]). *A topological space is k-space if it is a quotient image of some topological space Y.*

2 Some cardinal and topological properties of the n-permutation degree of a topological spaces

In this section we investigate some cardinal and topological properties of the n-permutation degree of a topological spaces.

Recall that a topological space is locally compact if for each $x \in X$ there exists a neighborhood U of x such that $[U]$ is a compact subspace of X [2].

Proposition 1. *Let X be a locally compact T_1-space, n positive integer and G subgroup of the permutation group S_n. Then $SP^n X, SP^n_G X, \exp_n X$ are k-spaces.*

Proof. Let X be a locally compact T_1-space. Then X^n is locally compact space for each $n \in N$. Spaces $SP^n X$ and $\exp_n X$ become a quotient image of the space X^n. So, $SP^n X, \exp_n X$ are k-spaces. Proposition is proved. □

Corollary 1. *Functors $SP^n, SP^n_G, \exp_n$ preserve any k-space.*

Proposition 2. *Let X be an infinite T_1-space, n positive integer and G subgroup of the permutation group S_n. Then $n\pi w\left(SP^n X\right) = n\pi w\left(X\right)$.*

Proof. In the work [3] it is proved that $d\left(SP^n X\right) = d\left(X\right), n \in N$. It is known that any dense set $M \subset X$ can be a π-net of this space. Thence, we have $n\pi w\left(SP^n X\right) = n\pi w\left(X\right)$. □

Corollary 2. *Let X be an infinite T_1-space, n positive integer and G subgroup of the permutation group S_n. Then $n\pi w\left(X\right) = n\pi w\left(SP^n X\right) = n\pi w\left(SP^n_G X\right) = n\pi w\left(SP^n_{G_1} X\right) = n\pi w\left(SP^n_{G_2} X\right) = n\pi w\left(\exp_n X\right) = n\pi w\left(\exp_\omega X\right) = n\pi w\left(\exp X\right)$.*

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3 Locally τ-density of hyperspaces

A set $A \subset X$ is dense in X if $[A] = X$. The density is defined as the smallest cardinal number of the form $|A|$, where A is a dense subset of X. This cardinal number is denoted by $d(X)$. A space X is said to be separable if $d(X) \leq \aleph_0$ [1].

We say that the weak density of the topological space is $\tau \geq \aleph_0$, if τ is the smallest cardinal number such that there exists a π-base coinciding with τ of centered systems of open sets, i.e. there is a π-base $B = \cup\{B_\alpha : \alpha \in A\}$, where B_α is a centered system of open sets for each $\alpha \in A$, $|A| = \tau$ [2].

Weak density of a topological space X is denoted by $wd(X)$. A topological space X is said to be weakly separable if $wd(X) = \aleph_0$ [3].

Definition 2 ([2])**.** *A topological space X is called a sequentially compact space if every sequence of points of X has a convergent subsequence.*

Theorem 1 ([2])**.** *If $f : X \to Y$ is a continuous mapping of a sequential compact space X onto a topological space Y, then Y is sequential compact.*

Theorem 2 ([2])**.** *The Cartesian product of countably many sequential compact spaces is sequentially compact.*

Proposition 3. *Let X be an infinite T_1-space, n positive integer and G subgroup of the permutation group S_n. Then spaces $SP^nX, SP^n_GX, \exp_nX$ are sequential compact.*

The proof of Proposition 3 follows from Theorems 1 and 2.

Corollary 3. *Functors $SP^n, SP^n_G, \exp_n$ preserve sequential compactness of any infinite T_1-space.*

Definition 3 ([4])**.** *A topological space X is locally separable at a point $x \in X$ if x has a separable neighborhood in X.*

A topological space is locally separable if it is locally separable at each point $x \in X$ [4].

Definition 4. *We say that the local density of a topological space X is τ at a point x if τ is the smallest cardinal number such that x has a neighborhood of density τ.*

The local density at a point x, is denoted by $ld(x)$. The local density of a topological space is defined as the supremum of all numbers $ld(x)$ for $x \in X$. The local density of X is denoted as following: $ld(X) = \sup\{ld(x) : x \in X\}$.

Proposition 4. *Let X be a space of local density τ and $f : X \to Y$ open continuous "onto" mapping. Then Y is space of local density τ.*

Proof. Since the map f is "onto", for every point $y \in Y$ the pre-image $f^{-1}(y)$ is nonempty in X. For each point $x \in f^{-1}(y)$ there exists a neighborhood Ox such that the density of Ox is τ. Since f is continuous, $f(Ox)$ is an open set in Y and contains the point y. It is known that the weak density is preserved under a continuous mapping, therefore $f(Ox)$ is weakly τ-dense in Y. □

Recall that a set is called a closed domain if it is the closure of its interior.

Theorem 3. *Assume that the locally density of X is τ and G is a subset of X. If G satisfies at least one of following conditions:*

a) G is open in X;

b) G is dense in X;

c) G is a closed domain in X;

then the locally density of G is τ.

Proof. a) Let G be a nonempty open subset of X. For any point $x \in G$ by the definition there exists a neighborhood $Ox \subset X$ such that Ox τ-dense. Then $Ox \bigcap G = O_1x$ is a nonempty open set in G, containing x. It is known that the density is hereditary with respect to open sets, therefore O_1x is τ-dense.

b) Let $M \subset X$ be a dense subset in X. Consider arbitrary point $y \in M$. Since X is locally τ-dense, there exists a neighborhood $Oy \subset X$ of y such that Oy is τ-dense. Consider $Oy \bigcap M = O_1y$. Then O_1y is a nonempty open subset in M. Besides, $O_1y \subseteq Oy$ and O_1y are dense in Oy. Since every dense subset of a dense space, is dense, O_1y is τ-dense.

c) Let G be closed domain in X. Then there exists a open set U such that $G = [U]$. In this case, by section a) U is locally τ-dense. We get arbitrary point $z \in G$ and τ-dense neighborhood $Oz \subset X$. Then $O_1z = Oz \bigcap G$ is a nonempty open set in G. Consider $V = Oz \bigcap U$, since every open subset of τ-dense space, is τ-dense, V is τ-dense. On the other hand, V is dense in O_1z. It is known that the density is hereditary with respect to dense set, therefore the neighborhood O_1z of z is τ-dense. □

Theorem 4. *The Cartesian product $X = \prod_{i=1}^{n} X_i$ is locally τ-dense iff X_i is locally τ-dense for each $i = 1, 2, \ldots, n$.*

Proof. Necessity. Suppose $X = \prod_{i=1}^{n} X_i$ and $p_i : X \to X_i$ is the projection of X onto X_i, i.e. $p_i(\{x\}) = x_i$, $x = \{x_k\} \in X$, $k = 1, 2, \ldots, n$. Since p_i $(i = 1, 2, \ldots, n)$ are continuous and are "onto", by proposition 3.8 we see that for each $i = 1, 2, \ldots, n$ the space X_i is locally τ-dense.

Sufficiency. Let $x = \{x_1, \ldots, x_n\}$ be an arbitrary point of X. Since X_i is locally τ-dense for each $i = 1, 2, \ldots, n$, there exist τ-dense neighborhoods Ox_i of x_i in X_i. Suppose $Ox = Ox_1 \times \ldots \times Ox_n$ is the Cartesian product of neighborhoods Ox_i. Then Ox is weakly τ-dense (proposition 1.1.11 (Wd6)) [4]. Thus, Ox is a τ-dense neighborhood of the point x. □

Theorem 5 ([5])**.** *Let X be an infinite T_1-space. $O\langle V_1, V_2, \ldots, V_k\rangle \subset O\langle U_1, U_2, \ldots, U_n\rangle$ iff for each $i = 1, 2, \ldots, n$ there exists $j = 1, 2, \ldots, k$ such that $V_j \subset U_i$.*

Theorem 6 ([5])**.** *Let X be an infinite T_1-space. $O\langle V_1, V_2, \ldots, V_k\rangle \subset O\langle U_1, U_2, \ldots, U_n\rangle$ iff for each $i = 1, 2, \ldots, n$ there exists $j = 1, 2, \ldots, k$ such that $V_j \subset U_i$.*

Theorem 7. *Let X be an infinite T_1-space. Then* $ld\,(X) = ld\,(\exp_n X) = ld\,(\exp_\omega X) = ld\,(\exp_c X)$.

Proof. First we show the equality $ld\,(X) = ld\,(\exp_n X)$. a) We shall show that $ld\,(\exp_n X) \leq ld\,(X)$. Assume that $ld\,(X) = \tau \geq \aleph_0$ and $F \in \exp_n X$ is an arbitrary element of $\exp_n X$. We shall prove that $ld\,(F) \leq \tau$. For convenience, suppose that the set $F = \{x_1, x_2, \ldots, x_n\}$ consists of exactly n distinct points. In this case, there exist neighborhoods $O_1x_1, O_2x_2, \ldots, O_nx_n$ of the point $\{x_1, x_2, \ldots, x_n\}$ such that $d\,(O_ix_i) \leq \tau$, $i = 1, 2, \ldots, n$. Let $M_1, M_2, \ldots, M_n$ be dense subsets in $O_1x_1, O_2x_2, \ldots, O_nx_n$, $i = 1, 2, \ldots, n$.

Consider finite subsets $M = \left\{ F_\alpha^i : F_\alpha^i \subset \bigcup\limits_{i=1}^{n} M_i, F_\alpha^i \cap M_i \neq \varnothing, i = 1, 2, \ldots, n \right\}$ of sets $M_1, M_2, \ldots, M_n$. It is obvious that $|M| \leq \tau$.

We shall show that M is dense in $O\,\langle U_1x_1, U_2x_2, \ldots, U_nx_n\rangle$. Let $O\,\langle V_1, V_2, \ldots, V_n\rangle \subset O\,\langle U_1x_1, U_2x_2, \ldots, U_nx_n\rangle$ be an arbitrary nonempty open subset of $\exp_n X$. We get an arbitrary element $E \in O\,\langle V_1, V_2, \ldots, V_n\rangle$, then $E \subset \bigcup\limits_{i=1}^{n} V_i$ and $E \cap V_i \neq \varnothing, i = 1, 2, \ldots, n$. Consequently, $E \in O\,\langle U_1x_1, U_2x_2, \ldots, U_nx_n\rangle$. This implies $E \subset \bigcup\limits_{i=1}^{n} U_ix_i$ and $E \cap U_ix_i \neq \varnothing, i = 1, 2, \ldots, n$. From theorem 3.11 [5] we see that for each $U_1x_1, U_2x_2, \ldots, U_nx_n$ there exist $V_1, V_2, \ldots, V_k$ such that $V_1 \subset U_1x_1, V_2 \subset U_2x_2, \ldots, V_k \subset U_nx_n$. Choose points $y_1 \in U_1 \cap M_1, y_2 \in U_2 \cap M_2, \ldots, y_n \in U_n \cap M_n$. Then $K = \{y_1, y_2, \ldots, y_n\} \in M$ and $K \in O\,\langle U_1, U_2, \ldots, U_n\rangle$. Therefore M is dense in $O\,\langle U_1x_1, U_2x_2, \ldots, U_nx_n\rangle$. Inequality a) is proved.

b) We shall prove the inequality $ld\,(\exp_n X) \geq ld\,(X)$. Suppose $ld\,(\exp_n X) = \tau \geq \aleph_0$. We shall show that $ld\,(X) \leq \tau$. Suppose $x \in X$. It is clear that $\{x\} \in \exp_n X$. In this case there exists a neighborhood $O\,\langle U\,\{x\}\rangle$ such that $d\,(O\,\langle U\,\{x\}\rangle) \leq \tau$. Consider a dense subset $S = \bigcup \{F_\alpha : \alpha \in A\}$ of $U\,\{x\}$ and $|S| \leq \tau$. From each F_α we take $x_\alpha \in F_\alpha$. Put $B = \{x_\alpha : x_\alpha \in F_\alpha\}$. It is obvious that $|B| \leq \tau$. We shall show that B is dense in $U\{x\}$. Let $G \subset U\,\{x\}$ be an arbitrary nonempty open subset of $U\{x\}$. Then $O\,\langle G\rangle$ is open in $O\,\langle U\,\{x\}\rangle$, i.e. $O\,\langle G\rangle \subset O\,\langle U\,\{x\}\rangle$. Since S is dense in $U\{x\}$, there exists an element such that $F_\alpha \in O\,\langle G\rangle$. It is clear that $F_\alpha \subset G$. By choosing of points $x_\alpha \in F_\alpha \subset G$. Thus the set B is dense in $U\{x\}$. We shall prove the inequality $ld\,(\exp_n X) \geq ld\,(X)$. From sections a) and b) we have $ld\,(X) = ld\,(\exp_n X)$. The equality $ld\,(X) = ld\,(\exp_\omega X)$ is proved analogously.

Now we shall prove the equality $ld\,(X) = ld\,(\exp_c X)$.

a) We shall show $ld\,(\exp_c X) \leq ld\,(X)$. Suppose $ld\,(X) = \tau \geq \aleph_0$. We take an arbitrary element $F \in \exp_c X$. Then $F \subset X$ is a compact subset of X. Since $ld\,(X) \leq \tau$, for each element $x \in F$ there is a neighborhood Ox such that $d\,(Ox) \leq \tau$. Assume that x runs over the set F, then the system $\{O_\alpha x_\alpha : x_\alpha \in F\}$ covers the set F. Since F is compact, there exist finite sets $O_1x_1, O_2x_2, \ldots, O_kx_k$ such that $\bigcup\limits_{i=1}^{k} O_ix_i \supseteq F$ and $d\,(O_ix_i) \leq \tau$ for each $i = 1, 2, \ldots, k$. It is clear that $O\,\langle O_1x_1, O_2x_2, \ldots, O_kx_k\rangle$ contains the compact F and we shall show that $d\,(O\,\langle O_1x_1, O_2x_2, \ldots, O_kx_k\rangle) \leq \tau$.

Indeed, let $M_1 = \left\{x_{\alpha_1}^1 : \alpha_1 \in A_1\right\}$, $M_2 = \left\{x_{\alpha_2}^2 : \alpha_2 \in A_2\right\}$, $\ldots$, $M_k =$

$\{x^k_{\alpha_k} : \alpha_k \in A_k\}$ be dense subsets of $O_1x_1, O_2x_2, \ldots, O_kx_k$. Consider finite $\left(x^1_{\alpha_1}, x^2_{\alpha_2}, \ldots, x^k_{\alpha_k}\right)$ combinations of sets $M_1, M_2, \ldots, M_k$. This set denote by $M = \left\{\left(x^1_{\alpha_1}, x^2_{\alpha_2}, \ldots, x^k_{\alpha_k}\right) : \alpha_i \in M_i, i = 1, 2, \ldots, k\right\}$, clearly, $|M| \leq \tau$. We shall prove that M is dense in $O\langle O_1x_1, O_2x_2, \ldots, O_kx_k\rangle$.

Indeed, assume that $O\langle U_1, U_2, \ldots, U_n\rangle \subset O\langle O_1x_1, O_2x_2, \ldots, O_kx_k\rangle$ is an arbitrary open set in $O\langle O_1x_1, O_2x_2, \ldots, O_kx_k\rangle$. Then $U_1, U_2, \ldots, U_n$ are open sets in X. From theorem 3.11 [5] it follows that for each set $O_ix_i, i = 1, 2, \ldots, k$ there exists $U_j, j = 1, 2, \ldots, n$ such that $U_j \subset O_ix_i$. For convenience, suppose $U_1 \subset O_1x_1, U_2 \subset O_2x_2, \ldots, U_n \subset O_kx_k$. Then $U_1 \cap M_1 \neq \varnothing, U_2 \cap M_2 \neq \varnothing, \ldots, U_n \cap M_k \neq \varnothing$. Choose a point from each intersection: $x_1 \in U_1 \cap M_1, x_2 \in U_2 \cap M_2, \ldots, x_n \in U_n \cap M_k$. These points denote by $F = \{x_1, x_2, \ldots, x_n\}$. It is clear that $F = \{x_1, x_2, \ldots, x_n\} \in O\langle U_1, U_2, \ldots, U_n\rangle \cap M \neq \varnothing$.

We shall prove that the set $M = \left\{\left(x^1_{\alpha_1}, x^2_{\alpha_2}, \ldots, x^k_{\alpha_k}\right) : \alpha_i \in M_i, i = 1, 2, \ldots, k\right\}$ is dense in $O\langle O_1x_1, O_2x_2, \ldots, O_kx_k\rangle$. Therefore $d\left(O\langle O_1x_1, O_2x_2, \ldots, O_kx_k\rangle\right) \leq \tau$. Inequality a) is proved.

b) Now we shall prove $ld(X) \leq ld(\exp_c X)$. Suppose $ld(\exp_c X) = \tau \geq \aleph_0$. We must prove $ld(X) \leq \tau$. Take an arbitrary point $x \in X$. Clearly, $\{x\} \in \exp_c X$, since $\{x\}$ is a compact set. Then there is a neighborhood $O\{x\}$ of x such that $d(\langle O\{x\}\rangle) \leq \tau$, where $\langle O\{x\}\rangle$ is an open set in $\exp_c X$ by definition of Vietoris topology. Since $x \in X$ is arbitrary, we have $ld(X) \leq \tau$. From a) and b) we obtain $ld(X) = ld(\exp_c X)$. □

Corollary 4. *Let X be an infinite compact T_1-space. Then $ld(X) = ld(\exp_n X) = ld(\exp_\omega X) = ld(\exp_c X) = ld(\exp X)$.*

Corollary 5. *Let X be an infinite compact T_1-space and $G_1 \subset G_2$ for subgroups G_1, G_2 of the permutation group S_n. Then $ld(X^n) = ld\left(SP^n_{G_1}X\right) = ld\left(SP^n_{G_2}X\right) = ld(SP^nX) = ld(\exp_n X)$.*

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