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CARLEMAN'S FORMULA FOR THE MATRIX UPPER HALF-PLANE

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Abstract

In this work the Carleman's formula for the matrix upper half-plane is obtained.

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Introduction

Integral representations of holomorphic functions play an important role in the classical theory of functions of one complex variable and in the multidimensional complex analysis. They solve the classic problem of recovery in the points of the D holomorphic function quite well, be having when approaching the boundary ∂D, from its values on ∂D or Shilov boundary S. Naturally, with this classical problem can be considered the following problem recover holomorphic function in D from its values on a set $M \subset \partial D$, which does not contain S. Of course, M will be a set of uniqueness for the class of holomorphic functions. The first result in this direction got T. Carleman in 1926 for the special domain $D \subset \mathbb{C}$. His idea of introducing a "quenching" function in the Cauchy integral formula Goluzin and Krylov developed in 1933 for simply connected flat domains (see [3]). Their method involved the construction of an auxiliary holomorphic function depending on the set M, it was possible to simply connected domains $D \subset \mathbb{C}$ but, in general case it is generally impossible for multiply connected domains in $\mathbb{C}$, or for domains in $\mathbb{C}^n,\ n > 1$.

1 Preliminaries

Matrix unit disc (classical domain of the first type according to the classification of Cartan) is defined as a set

$$\tau = \left\{ Z \in \mathbb{C}[m \times m] : ZZ^* < I \right\},$$

where $Z^* = \overline{Z}'$ is adjoint and transpose matrix of Z, and $ZZ^* < I$ (I is the identity $[m \times m]$-matrix) means that the Hermitian matrix $I - ZZ^*$ is positively defined, i.e. all its eigenvalues are positive. The boundary of τ consist of

$$\partial\tau = \left\{ Z \in \mathbb{C}[m \times m] : \det(I - ZZ^*) = 0, ZZ^* \leq I \right\},$$

i.e. consist of the set of matrices Z, for which the matrix $I - ZZ^*$ is a nonnegatively defined, but not positively defined Hermitian matrix (its eigenvalues are non-negative and at least one of them is zero). The boundary includes the set

$$S(\tau) = \{Z \in \mathbb{C}[m \times m] : ZZ^* = I\},$$

which is called *skeleton* of τ (note that $S(\tau)$ is the Shilov boundary for τ). It is clear that $S(\tau)$ the set of all unitary $[m \times m]$-matrix (the set of unitary matrices of order m is usually referred to the $U(m)$). It should be noted that the set of matrices $\{Z : \det(I - ZZ^*) = 0\}$ contains a limited component, distinguished by the condition $ZZ^* \leq I$, and unlimited for $ZZ^* \geq I$. These components intersect in the skeleton $S(\tau)$.

2 Carleman's formula

Let τ be a matrix unit disc, and $S(\tau)$ is its skeleton (the Shilov boundary), the set $M \subset S(\tau)$ and $\mu(M) > 0$, where μ is normalized Lebesgue measure on $S(\tau)$.

We parametrize $S(\tau)$ in the following way: $U = e^{i\phi}u$, $0 \leq \phi \leq 2\pi$, $u \in SU(m)$, where $SU(m)$ is a group of special unitary matrices, i.e. $\det U = 1$. Since $\det U = e^{im\phi} \det u = e^{im\phi}$, the set $\{U : U = \lambda u, |\lambda| = 1\}$, $u \in SU(m)$ intersects the set of elements of the group $SU(m)$ at exactly m roots of unity $e^{im\phi} = 1$.

Lemma 1 (see. [1]). *Haar's measure $d\mu$ of the manifold $S(\tau)$ can be written as*

$$d\mu = h(u)d\phi d\mu_0(u),$$

where $d\mu_0$ is normalized Lebesgue measure on $SU(m)$, and h is a smooth positive function on $SU(m)$.

We introduce the set

$$M_{0,u} = \left\{U : U \in M,\, U = \lambda u,\ \lambda = e^{i\phi},\ 0 \leq \phi \leq 2\pi\right\},\, u \in SU(m),$$

$$M_0' = \{u :\ u \in SU(m)\,,\, m_1 M_{0,u} > 0\}\,,$$

where m_1 is Lebesgue measure.

According to Fubini's theorem $\mu_0(M_0') > 0$. Let

$$\psi_0(U) = \frac{1}{2\pi i} \int\limits_{M_{0,u}} \frac{\eta + \lambda}{\eta - \lambda} \frac{d\eta}{\eta}, \quad \phi_0 = \exp \psi_0.$$

Lemma 2 (see [2]). *Let $f \in H^1(\tau)$ ($H^1(\tau)-$ is Hardy's class). Then the following formula*

$$f(0) = \frac{m}{\int\limits_{M_0'} d\mu_1} \lim_{j\to\infty} \int\limits_{M} f(U) \left[\frac{\phi_0(U)}{\phi_0(0)}\right]^j d\mu_U$$

is true.

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Let $\varphi_A(Z)$ be an automorphism of τ, which transforms point A to 0. Let

$$\mu_A(K) = \mu_1({\varphi_A}^{-1})(K)), \;\; M_{A,\omega} = \{U : U \in M, \, U =$$

$$= {\varphi_A}^{-1}\left(\lambda {\varphi_A}^{-1}(\omega)\right)\}, \;\; |\lambda| = 1, \;\; \omega \in S_A = \varphi_A(SU(m)),$$

$$M_A^{'} = \{\omega : \, \omega \in S_A \, , \, m_1 M_{A,\omega} > 0\},$$

$$\psi_A(U) = \frac{1}{2\pi i} \int\limits_{M_{A,\omega}} \frac{\eta + \lambda}{\eta - \lambda} \frac{d\eta}{\eta}, \quad \varphi_A = \exp \psi_A$$

(ψ_A depends on U, because λ and ω are functions of U).

Theorem 1 (see. [2]). *Let $f \in H^1(\tau)$. Then for any point $A \in \tau$ the following Carleman's formula holds*

$$f(A) = \frac{m}{\int\limits_{M_A^{'}} d\mu_A} \lim_{j \to \infty} \int\limits_{M} f(U) \left[\frac{\varphi_A(U)}{\varphi_A(A)}\right]^j H(A, \overline{U}) d\mu_U, \tag{1}$$

where $H(A, \overline{U})$ is Cauchy kernel for the matrix unit disk.

Matrix semi half-plane is a domain (see [1].)

$$D = \{W \in \mathbb{C}[m \times m] : ImW > 0\},$$

where $W = \|w_{jk}\|$, $(j, k = 1, \ldots, m)$ is square matrix of order m, whose elements are complex numbers of the $\mathbb{C}$, here ImW is defined as

$$ImW = \frac{1}{2i}(W - W^*),$$

where W^* is a conjugate and transpose matrix of W. Obviously the matrix ImW is Hermitian: its elements $h_{jk} = \frac{1}{2i}(w_{jk} - \bar{w}_{kj})$ satisfy the conditions $\bar{h}_{jk} = h_{kj}$, and, in particular, $h_{jj} = Imw_{jj}$ are real. The inequality $H > 0$ for Hermitian matrix H means that it is positively defined, i.e. all its eigenvalues are positive.

The boundary ∂D of the domain D is matrix W for which ImW is non-negatively but not positively defined Hermitian matrix (its eigenvalues are non-negative and at least one of them is zero). Since the vanishing eigenvalues of the Hermitian matrix is expressed as a real analytic equation, then ∂D consists of pieces of real analytic surfaces of dimension $2m^2 - 1$.

On the ∂D we define a set

$$\Gamma = \{W \in \mathbb{C}[m \times m] : ImW = 0\},$$

which is called a skeleton of the domain D. It consists of all Hermitian matrices of order m. Hermit's condition expressed via m^2 independent equations, so the real dimension of Γ is m^2.

Let Φ is the transformation

$$W = \Phi(Z) = i(I + Z)(I - Z)^{-1}, \tag{2}$$

which biholomorphically maps τ to D, while $S(\tau)$ goes to Γ (see [1]).

With the transformation Φ and automorphism Φ_A of the matrix unit disk, which transforms the point $A \in \tau$ in 0 (0 is a zero matrix of order m), we define following transformation

$$\Psi_B = \Phi \circ \Phi_A \circ \Phi^{-1},\ B = \Phi(A),$$

which is the automorphism of the domain D, transforming point B of D to the point iI.

Let $\dot{U}$ – element of the volume in $S(\tau)$, a $\dot{V}$ – element of the volume in Γ. In [4, §3.1] is proved the following relation between $\dot{U}$ and $\dot{V}$ under the mapping Φ:

$$\dot{U} = 2^{m^2}(\det(V^2 + I))^{-m}\dot{V}, \tag{3}$$

where $V \in \Gamma$. Since $V^* = V$ and

$$\det(V^2 + I) = \det(V - iI)\det(V + iI) =$$
$$= \overline{\det(V + iI)}\det(V + iI) = |\det(V + iI)|^2,$$

(3) can be written as

$$\dot{U} = 2^{m^2}|\det(V + iI)|^{-2m}\dot{V}. \tag{4}$$

The class of holomorphic functions in D we denote as $A(D)$.

For consideration of multidimensional analogues of Carleman's formulas it is desirable to extend the class of functions for which these formulas are true in the upper matrix half plane D. Let $f \in A(D)$. Note that $f(i(I + Z)(I - Z)^{-1}) \in H^1(\tau)$ if and only if ([5], page 147)

$$f(W)\det^{-2}(W + iI) \in H^1(D). \tag{5}$$

Theorem 2. *If the function $f \in A(D)$ satisfies the condition (5) and the set $\tilde{M} \in \partial D$ has positive Lebesgue measure, then the following Carleman's formula is true*,

$$f(W) = \frac{\det^m(W + iI)}{i^{m^2}} \times \lim_{j\to\infty}\int\limits_{\tilde{M}} f(V)\left[\frac{\tilde{\varphi}(V)}{\tilde{\varphi}(W)}\right]^j \frac{d\mu_V}{\det^m(V^* - W)\det^m(V + iI)}, \tag{6}$$

where the limit is uniform on compact subsets of ∂D, and $V \in \tilde{M}$.

Proof. Let $F(Z) = f(i(I + Z)(I - Z)^{-1})$, then $f(Z) \in H^1(\tau)$ and by Theorem 1 for it Carleman's formula holds

$$F(Z) = \lim_{j\to\infty}\int\limits_{M} F(U)\left[\frac{\varphi(U)}{\varphi(Z)}\right]^j \frac{d\mu_U}{\det^m(I - ZU^*)},$$

where M is image of $\tilde{M}$ under the mapping $Z = (W + iI)(W - iI)^{-1}$, of the matrix upper half plane to the matrix unit disc.

Next, we consider the inverse mapping of (2)

$$Z = (W + iI)^{-1}(W - iI),\ \ U = (V + iI)^{-1}(V - iI),$$

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and make the following calculations:

$$I - ZU^* = I - (W + iI)^{-1}(W - iI)(V^* + iI)(V^* - iI)^{-1} =$$

$$= (W + iI)^{-1}\left[(W + iI)(V^* - iI) - (W - iI)(V^* + iI)\right](V^* - iI)^{-1} =$$

$$= (W + iI)^{-1}\left[WV^* - iW + iV^* + I - WV^* - iW - iV^* - I\right](V^* - iI)^{-1} =$$

$$= 2i(W + iI)^{-1}\left[V^* - W\right](V^* - iI)^{-1},$$

and the condition (4) holds

$$d\mu_U = 2^{m^2}\left|\det(V + iI)\right|^{-2m} d\mu_V.$$

Calculations show, that

$$\frac{d\mu_U}{\det^m(I - ZU^*)} = \frac{\det^m(W + iI)\det^m(V^* - iI)}{(2i)^{m^2}\det^m(V^* - W)} \cdot \frac{2^{m^2} d\mu_V}{\left|\det^m(V + iI)\right|^2} =$$

$$= \frac{\det^m(W + iI)}{i^{m^2}\det^m(V^* - W)\det^m(V + iI)} d\mu_V.$$

Next, φ plays the role of $\tilde{\varphi}$ for the set of M. According to M. A. Lavryentyev's theorem the set M has also positive Lebesgue measure such that the harmonic measure of M goes into harmonic measure of $\tilde{M}$, therefore, φ goes to $\tilde{\varphi}$, and we get the formula (6).

Remark. If instead of square matrices of order m we consider symmetric matrices of order m, the domain τ turns into a classic domain of type 2, according to the Cartan classification. In this case the considered domain D is called the "Siegel matrix half-plane". Theorem 2 for this domain has the following form

Theorem 3. *If the function $f \in A(D)$ satisfies the condition (5) and the set $\tilde{M} \in \partial D$ has positive Lebesgue measure, then the following formula is true*

$$f(W) = \frac{\det^{\frac{m+1}{2}}(W + iI)}{i^{\left(\frac{m+1}{2}\right)^2}} \times$$

$$\lim_{j\to\infty}\int_{\tilde{M}} f(V)\left[\frac{\tilde{\varphi}(V)}{\tilde{\varphi}(W)}\right]^j \frac{d\mu_V}{\det^{\frac{m+1}{2}}(V^* - W)\det^{\frac{m+1}{2}}(V + iI)}, \quad (7)$$

where the limit is uniform on compact subsets of ∂D.

From the proved formula (6), in particular, for $m = 1$ the known Carleman's formula for the upper half-plane is deduced.

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