Positive fixed points of Lyapunov integral operators and Gibbs measures

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POSITIVE FIXED POINTS OF LYAPUNOV INTEGRAL OPERATORS AND GIBBS MEASURES

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Abstract
In this paper it is found fixed points of Lyapunov integral equation and considered the connections between Gibbs measures for four competing interactions of models with uncountable (i.e. \([0, 1]\)) set of spin values on the Cayley tree of order two.

Keywords: Lyupanov integral operator, fixed points, Cayley tree, Gibbs measure, translation-invariant Gibbs measure.

Mathematics Subject Classification (2010): 46G12, 45P05

1 Preliminaries

A Cayley tree \(\Gamma^k = (V, L)\) of order \(k \in \mathbb{N}\) is an infinite homogeneous tree, i.e., a graph without cycles, with exactly \(k+1\) edges incident to each vertices. Here \(V\) is the set of vertices and \(L\) that of edges (arcs). Two vertices \(x\) and \(y\) are called nearest neighbors if there exists an edge \(l \in L\) connecting them. We will use the notation \(l = \langle x, y \rangle\). The distance \(d(x, y), x, y \in V\) on the Cayley tree is defined by the formula

\[
d(x, y) = \min\{d\mid x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V \text{ such that the pairs } < x_0, x_1 >, \ldots, < x_{d-1}, x_d > \text{ are neighboring vertices}\}.
\]

Let \(x^0 \in V\) be a fixed and we set

\[
W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \{x \in V \mid d(x, x^0) \leq n\},
\]

\[
L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\},
\]

The set of the direct successors of \(x\) is denoted by \(S(x)\), i.e.

\[
S(x) = \{y \in W_{n+1} \mid d(x, y) = 1\}, \quad x \in W_n.
\]

We observe that for any vertex \(x \neq x^0\), \(x\) has \(k\) direct successors and \(x^0\) has \(k+1\). The vertices \(x\) and \(y\) are called second neighbor which is denoted by \(> x, y \,<\), if there exist a vertex \(z \in V\) such that \(x, z\) and \(y, z\) are nearest neighbors. We will consider only second neighbors \(> x, y \,<\), for which there exist \(n\) such that \(x, y \in W_n\). Three vertices \(x, y\) and \(z\) are called a triple of neighbors and they are denoted by \(< x, y, z >\), if \(< x, y >\), \,< y, z >\) are nearest neighbors and \(x, z \in W_n, y \in W_{n-1}\), for some \(n \in \mathbb{N}\).
Now we consider models with four competing interactions where the spin takes values in the set $[0, 1]$. For some set $A \subset V$ an arbitrary function $\sigma_A : A \to [0, 1]$ is called a configuration and the set of all configurations on $A$ we denote by $\Omega_A = [0, 1]^A$. Let $\sigma(\cdot)$ belong to $\Omega_V = \Omega$ and $\xi : (t, u, v) \in [0, 1]^3 \to \xi(t, u, v) \in R$, $\xi_i : (u, v) \in [0, 1]^2 \to \xi_i(u, v) \in R$, $i \in \{2, 3\}$ are given bounded, measurable functions. Then we consider the model with four competing interactions on the Cayley tree which is defined by following Hamiltonian

$$H(\sigma) = -J_3 \sum_{<x,y,z>} \xi_1(\sigma(x), \sigma(y), \sigma(z)) - J \sum_{>x,y<} \xi_2(\sigma(x), \sigma(z)) - J_1 \sum_{<x,y>} \xi_3(\sigma(x), \sigma(y)) - \alpha \sum_{x \in V} \sigma(x),$$

(1)

where the sum in the first term ranges all triples of neighbors, the second sum ranges all second neighbors, the third sum ranges all nearest neighbors and $J, J_1, J_3, \alpha \in R \setminus \{0\}$. Let $h : [0, 1] \times V \setminus \{x_0\} \to R$ and $|h(t, x)| = |h_t(x)| < C$ where $x_0$ is a root of Cayley tree and $C$ is a constant which does not depend on $t$. For some $n \in \mathbb{N}$, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and $Z_n$ is the corresponding partition function we consider the probability distribution $\mu^{(n)}$ on $\Omega_{V_n}$ defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp \left( -\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x} \right),$$

(2)

$$Z_n = \int \ldots \int_{\Omega_n^{(p)}} \exp \left( -\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma_n(x), x} \right) \lambda_{V_n-1}^{(p)}(d\sigma_n),$$

(3)

where

$$\frac{\Omega_n \times \Omega_n \times \ldots \times \Omega_n}{3^{2^{p-1}}} = \Omega_n^{(p)}, \quad \frac{\lambda_n \times \lambda_n \times \ldots \times \lambda_n}{3^{2^{p-1}}} = \lambda_n^{(p)}, \quad n, p \in \mathbb{N},$$

Let $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of $\sigma_{n-1}$ and $\omega_n$. For $n \in \mathbb{N}$ we say that the probability distributions $\mu^{(n)}$ are compatible if $\mu^{(n)}$ satisfies the following condition:

$$\int \int_{\Omega_{W_n} \times \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n)(\lambda_n \times \lambda_n)(d\omega_n) = \mu^{(n-1)}(\sigma_{n-1}).$$

(4)

By Kolmogorov’s sigma extension theorem there exists a unique measure $\mu$ on $\Omega_V$ such that, for any $n$ and $\sigma_n \in \Omega_{V_n}$, $\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n)$. The measure $\mu$ is called splitting Gibbs measure corresponding to Hamiltonian (1) and function $x \mapsto h_x$, $x \neq x_0$ (see [1], [2], [6]).

Denote

$$K(u, t, v) = \exp \left\{ J_3 \beta \xi_1(t, u, v) + J \beta \xi_2(u, v) + J_1 \beta (\xi_3(t, u) + \xi_3(t, v)) + \alpha \beta (u + v) \right\},$$

(5)
and
\[ f(t, x) = \exp(h_{t,x} - h_{0,x}), \quad (t, u, v) \in [0, 1]^3, \quad x \in V \setminus \{x^0\}. \]

The following statement describes conditions on \( h_x \) guaranteeing compatibility of the corresponding distributions \( \mu^{(n)}(\sigma_n) \).

**Proposition 1** [5]. The measure \( \mu^{(n)}(\sigma_n) \), \( n = 1, 2, \ldots \) satisfies the consistency condition (4) iff for any \( x \in V \setminus \{x^0\} \) the following equation holds:

\[ f(t, x) = \prod_{y, z \in S(x)} \frac{\int_0^1 \int_0^1 K(t, u, v) f(u, y) f(v, z) dudv}{\int_0^1 \int_0^1 K(0, u, v) f(u, y) f(v, z) dudv}, \tag{6} \]

where \( S(x) = \{y, z\} \), \( < y, x, z > \) is a ternary neighbor.

## 2 Main results

Now we prove that there exist at least one fixed point of Lyapunov integral equation, namely there is a splitting Gibbs measure corresponding to Hamiltonian (1).

**Proposition 2.** Let \( J_3 = J = \alpha = 0 \) and \( J_1 \neq 0 \). Then (6) is equivalent to

\[ f(t, x) = \prod_{y \in S(x)} \frac{\int_0^1 \int_0^1 \exp \{J_1 \beta \xi_3(t, u)\} f(u, y) f(v, z) dudv}{\int_0^1 \int_0^1 \exp \{J_1 \beta \xi_3(0, u)\} f(u, y) f(v, z) dudv}, \tag{7} \]

where \( f(t, x) = \exp(h_{t,x} - h_{0,x}) \), \( t \in [0, 1] \), \( x \in V \).

**Proof.** For \( J_3 = J = \alpha = 0 \) and \( J_1 \neq 0 \) one get \( K(t, u, v) = \exp \{J_1 \beta (\xi_3(u, t) + \xi_3(v, t))\} \). Then (6) can be written as

\[ f(t, x) = \frac{\prod_{y, z \in S(x)} \int_0^1 \exp \{J_1 \beta \xi_3(t, u)\} f(u, y) f(v, z) dudv}{\int_0^1 \int_0^1 \exp \{J_1 \beta \xi_3(0, u)\} f(u, y) f(v, z) dudv} = \]

\[ \prod_{y, z \in S(x)} \frac{\int_0^1 \exp \{J_1 \beta \xi_3(t, u)\} f(u, y) f(v, z) dudv}{\int_0^1 \exp \{J_1 \beta \xi_3(0, u)\} f(u, y) f(v, z) dudv}. \tag{8} \]

Since \( > y, z \in S(x) \) equation (8) is equivalent to (7).

Now we consider the case \( J_3 \neq 0, J = J_1 = \alpha = 0 \) for the model (1) in the class of translational-invariant functions \( f(t, x) \) i.e \( f(t, x) = f(t) \), for any \( x \in V \). For such functions equation (1) can be written as

\[ f(t) = \frac{\int_0^1 \int_0^1 K(t, u, v) f(u) f(v) dudv}{\int_0^1 \int_0^1 K(0, u, v) f(u) f(v) dudv}, \tag{9} \]

where \( K(t, u, v) = \exp \{J_3 \beta \xi_1(t, u, v) + J_2 \beta \xi_2(u, v) + J_1 \beta (\xi_3(t, u) + \xi_3(v, t)) + \alpha \beta (u + v)\} \), \( f(t) > 0, \quad t, u \in [0, 1] \).

We shall find positive continuous solutions to (9) i.e. such that \( f \in C^+[0, 1] = \{f \in C[0, 1] : f(x) \geq 0\} \).
Define a nonlinear operator $H$ on the cone of positive continuous functions on $[0, 1]$:

$$(Hf)(t) = \frac{\int_0^1 \int_0^1 K(t, s, u)f(s)f(u)dsdu}{\int_0^1 \int_0^1 K(0, s, u)f(s)f(u)dsdu}.$$ 

We’ll study the existence of positive fixed points for the nonlinear operator $H$ (i.e., solutions of the equation (9)). Put $C^+_0[0, 1] = C^+[0, 1] \setminus \{\theta \equiv 0\}$. Then the set $C^+[0, 1]$ is the cone of positive continuous functions on $[0, 1]$.

We define the Lyapunov integral operator $\mathcal{L}$ on $C[0, 1]$ by the equality (see [3])

$$\mathcal{L}f(t) = \int_0^1 K(t, s, u)f(s)f(u)dsdu.$$ 

Put

$$\mathcal{M}_0 = \{f \in C^+[0, 1] : f(0) = 1\}.$$ 

**Lemma 3.** The equation $Hf = f$ has a nontrivial positive solution iff the Lyapunov equation $\mathcal{L}g = g$ has a nontrivial positive solution.

**Proof.** At first we shall prove that the equation

$$Hf = f, \ f \in C^+_0[0, 1] \tag{10}$$

has a positive solution iff the Lyapunov equation

$$\mathcal{L}g = \lambda g, \ g \in C^+[0, 1] \tag{11}$$

has a positive solution in $\mathcal{M}_0$ for some $\lambda > 0$.

Let $\lambda_0$ be a positive eigenvalue of the Lyapunov operator $\mathcal{L}$. Then there exists $f_0 \in C^+_0[0, 1]$ such that $\mathcal{L}f_0 = \lambda_0 f_0$. Take $\lambda \in (0, +\infty)$, $\lambda \neq \lambda_0$. Define the function $h_0(t) \in C^+_0[0, 1]$ by $h_0(t) = \frac{\lambda}{\lambda_0} f_0(t)$, $t \in [0, 1]$. Then $\mathcal{L}h_0 = \lambda h_0$, i.e., the number $\lambda$ is an eigenvalue of Lyapunov operator $\mathcal{L}$ corresponding the eigenfunction $h_0(t)$. It’s easy to check that if the number $\lambda_0 > 0$ is an eigenvalue of the operator $\mathcal{L}$, then an arbitrary positive number is eigenvalue of the operator $\mathcal{L}$. Now we shall prove the lemma. Let equation (11) holds then the function $\frac{1}{\lambda} g(t)$ be a fixed point of the operator $\mathcal{L}$. Analogously, since $H$ is non-linear operator we can correspond to the fixed point if there exist any eigenvector.

**Proposition 4.** The equation

$$\mathcal{L}f = \lambda f, \ \lambda > 0 \tag{12}$$

has at least one solution in $C^+_0[0, 1]$.

**Proof.** Clearly, that the Lyapunov operator $\mathcal{L}$ is a compact on the cone $C^+[0, 1]$. By the other hand we have

$$\mathcal{L}f(t) \geq m \left(\int_0^1 f(s)ds\right)^2,$$

for all $f \in C^+[0, 1]$, where $m = \min K(t, s, u) > 0$. 

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Put $\Gamma = \{ f : \| f \| = r, f \in C[0, 1] \}$. We define the set $\Gamma_+$ by

$$\Gamma_+ = \Gamma \cap C^+[0, 1].$$

Then we obtain

$$\inf_{f \in \Gamma_+} \| Lf \| > 0.$$ 

Then by Schauder’s theorem (see [4], p. 20) there exists a number $\lambda_0 > 0$ and a function $f_0 \in \Gamma_+$ such that $Lf_0 = \lambda_0 f_0$.

Denote by $N_{fix,p}(H)$ and $N_{fix,p}(L)$ are the set of positive numbers of nontrivial positive fixed points of the operators $N_{fix,p}(H)$ and $N_{fix,p}(L)$, respectively. By Lemma 3 and Proposition 4 we can conclude that:

**Proposition 5.** (a) The equation (10) has at least one solution in $C^+_0[0, 1]$.
(b) The equality $N_{fix,p}(H) = N_{fix,p}(L)$ is hold.

From Proposition 1 and the last Proposition we get the following theorem.

**Theorem 6.** The set of splitting Gibbs measures corresponding to Hamiltonian (1) is non-empty.

**References**


