Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences

Volume 1 | Issue 1 Article 9

3-25-2018

Positive fixed points of Lyapunov integral operators and Gibbs measures

Farkhod Haydarov
National University of Uzbekistan, haydarov_imc@mail.ru

POSITIVE FIXED POINTS OF LYAPUNOV INTEGRAL OPERATORS AND GIBBS MEASURES

Haydarov F. H.
National University of Uzbekistan, Tashkent, Uzbekistan
e-mail: haydarov_imc@mail.ru

Abstract

In this paper it is found fixed points of Lyapunov integral equation and considered the connections between Gibbs measures for four competing interactions of models with uncountable (i.e. $[0,1]$) set of spin values on the Cayley tree of order two.

Keywords: Lyupanov integral operator, fixed points, Cayley tree, Gibbs measure, translation-invariant Gibbs measure.

Mathematics Subject Classification (2010): 46G12, 45P05

1 Preliminaries

A Cayley tree $\Gamma^k = (V, L)$ of order $k \in \mathbb{N}$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k+1$ edges incident to each vertices. Here V is the set of vertices and L that of edges (arcs). Two vertices x and y are called nearest neighbors if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y\rangle$. The distance $d(x,y), x, y \in V$ on the Cayley tree is defined by the formula

$$d(x,y) = \min\{d|\ x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V \ \textit{such that the pairs}$$

$$< x_0, x_1 >, \dots, < x_{d-1}, x_d > \textit{are neighboring vertices}\}.$$

Let $x^0 \in V$ be a fixed and we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \{x \in V \mid d(x, x^0) \le n\},$$

$$L_n = \{l =< x, y >\in L \mid x, y \in V_n\},$$

The set of the direct successors of x is denoted by $S(x)$, i.e.

$$S(x) = \{y \in W_{n+1}|\ d(x,y) = 1\},\ x \in W_n.$$

We observe that for any vertex $x \neq x^0,\ x$ has k direct successors and x^0 has $k+1$. The vertices x and y are called second neighbor which is denoted by $> x, y <$, if there exist a vertex $z \in V$ such that $x,\ z$ and $y,\ z$ are nearest neighbors. We will consider only second neighbors $> x, y <$, for which there exist n such that $x, y \in W_n$. Three vertices $x,\ y$ and z are called a triple of neighbors and they are denoted by $< x, y, z >$, if $< x, y >,\ < y, z >$ are nearest neighbors and $x,\ z \in W_n,\ y \in W_{n-1}$, for some $n \in \mathbb{N}$.

Now we consider models with four competing interactions where the spin takes values in the set $[0,1]$. For some set $A \subset V$ an arbitrary function $\sigma_A : A \to [0,1]$ is called a configuration and the set of all configurations on A we denote by $\Omega_A = [0,1]^A$. Let $\sigma(\cdot)$ belong to $\Omega_V = \Omega$ and $\xi_1 : (t,u,v) \in [0,1]^3 \to \xi_1(t,u,v) \in R$, $\xi_i : (u,v) \in [0,1]^2 \to \xi_i(u,v) \in R$, $i \in \{2,3\}$ are given bounded, measurable functions. Then we consider the model with four competing interactions on the Cayley tree which is defined by following Hamiltonian

$$H(\sigma) = -J_3 \sum_{<x,y,z>} \xi_1(\sigma(x),\sigma(y),\sigma(z)) - J \sum_{>x,y<} \xi_2(\sigma(x),\sigma(z))$$
$$-J_1 \sum_{<x,y>} \xi_3(\sigma(x),\sigma(y)) - \alpha \sum_{x\in V} \sigma(x), \tag{1}$$

where the sum in the first term ranges all triples of neighbors, the second sum ranges all second neighbors, the third sum ranges all nearest neighbors and $J, J_1, J_3, \alpha \in R \setminus \{0\}$. Let $h : [0,1] \times V \setminus \{x^0\} \to \mathbb{R}$ and $|h(t,x)| = |h_{t,x}| < C$ where x_0 is a root of Cayley tree and C is a constant which does not depend on t. For some $n \in \mathbb{N}$, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and Z_n is the corresponding partition function we consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x\in W_n} h_{\sigma(x),x}\right), \tag{2}$$

$$Z_n = \underset{\Omega^{(p)}_{V_{n-1}}}{\int \ldots \int} \exp\left(-\beta H(\widetilde{\sigma}_n) + \sum_{x\in W_n} h_{\widetilde{\sigma}(x),x}\right) \lambda^{(p)}_{V_{n-1}}(d\widetilde{\sigma}_n), \tag{3}$$

where

$$\underbrace{\Omega_{W_n} \times \Omega_{W_n} \times \ldots \times \Omega_{W_n}}_{3\cdot 2^{p-1}} = \Omega^{(p)}_{W_n}, \quad \underbrace{\lambda_{W_n} \times \lambda_{W_n} \times \ldots \times \lambda_{W_n}}_{3\cdot 2^{p-1}} = \lambda^{(p)}_{W_n}, \; n,p \in \mathbb{N},$$

Let $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of σ_{n-1} and ω_n. For $n \in \mathbb{N}$ we say that the probability distributions $\mu^{(n)}$ are compatible if $\mu^{(n)}$ satisfies the following condition:

$$\iint_{\Omega_{W_n}\times\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n)(\lambda_{W_n} \times \lambda_{W_n})(d\omega_n) = \mu^{(n-1)}(\sigma_{n-1}). \tag{4}$$

By Kolmogorov's sigma extension theorem there exists a unique measure μ on Ω_V such that, for any n and $\sigma_n \in \Omega_{V_n}$, $\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n)$. The measure μ is called *splitting Gibbs measure* corresponding to Hamiltonian (1) and function $x \mapsto h_x$, $x \neq x^0$ (see [1], [2], [6]).

Denote

$$K(u,t,v) = \exp\left\{J_3\beta\xi_1(t,u,v) + J\beta\xi_2(u,v) + J_1\beta(\xi_3(t,u) + \xi_3(t,v)) + \alpha\beta(u+v)\right\}, \tag{5}$$

and

$$f(t,x)=\exp(h_{t,x}-h_{0,x}),\quad (t,u,v)\in[0,1]^3,\ x\in V\setminus\{x^0\}.$$

The following statement describes conditions on h_x guaranteeing compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

Proposition 1 [5]. *The measure $\mu^{(n)}(\sigma_n)$, $n=1,2,\dots$ satisfies the consistency condition (4) iff for any $x\in V\setminus\{x^0\}$ the following equation holds:*

$$f(t,x)=\prod_{>y,z<\in S(x)}\frac{\int_0^1\int_0^1 K(t,u,v)f(u,y)f(v,z)dudv}{\int_0^1\int_0^1 K(0,u,v)f(u,y)f(v,z)dudv},\tag{6}$$

where $S(x)=\{y,z\}$, $<y,x,z>$ is a ternary neighbor.

2 Main results

Now we prove that there exist at least one fixed point of Lyapunov integral equation, namely there is a splitting Gibbs measure corresponding to Hamiltonian (1).

Proposition 2. *Let $J_3=J=\alpha=0$ and $J_1\neq 0$. Then (6) is equivalent to*

$$f(t,x)=\prod_{y\in S(x)}\frac{\int_0^1\exp\{J_1\beta\xi_3(t,u)\}f(u,y)du}{\int_0^1\exp\{J_1\beta\xi_3(0,u)\}f(u,y)du},\tag{7}$$

where $f(t,x)=\exp(h_{t,x}-h_{0,x})$, $t\in[0,1]$, $x\in V$.

Proof. For $J_3=J=\alpha=0$ and $J_1\neq 0$ one get $K(t,u,v)=\exp\{J_1\beta\left(\xi_3(u,t)+\xi_3(v,t)\right)\}$. Then (6) can be written as

$$f(t,x)=\prod_{>y,z<\in S(x)}\frac{\int_0^1\int_0^1\exp\{J_1\beta\left(\xi_3(t,u)+\xi_3(t,v)\right)\}f(u,y)f(v,z)dudv}{\int_0^1\int_0^1\exp\{J_1\beta\left(\xi_3(0,u)+\xi_3(0,v)\right)\}f(u,y)f(v,z)dudv}=$$

$$\prod_{>y,z<\in S(x)}\frac{\int_0^1\exp\{J_1\beta\xi_3(t,u)\}f(u,y)du\cdot\int_0^1\exp\{J_1\beta\xi_3(t,v)\}f(v,z)dv}{\int_0^1\exp\{J_1\beta\xi_3(0,u)\}f(u,y)du\cdot\int_0^1\exp\{J_1\beta\xi_3(0,v)\}f(v,z)dv}.\tag{8}$$

Since $>y,z<=S(x)$ equation (8) is equivalent to (7).

Now we consider the case $J_3\neq 0,\ J=J_1=\alpha=0$ for the model (1) in the class of translational-invariant functions $f(t,x)$ i.e $f(t,x)=f(t)$, for any $x\in V$. For such functions equation (1) can be written as

$$f(t)=\frac{\int_0^1\int_0^1 K(t,u,v)f(u)f(v)dudv}{\int_0^1\int_0^1 K(0,u,v)f(u)f(v)dudv},\tag{9}$$

where $K(t,u,v)=\exp\{J_3\beta\xi_1(t,u,v)+J\beta\xi_2(u,v)+J_1\beta\left(\xi_3(t,u)+\xi_3(t,v)\right)+\alpha\beta(u+v)\}$, $f(t)>0,\ t,u\in[0,1]$.

We shall find positive continuous solutions to (9) i.e. such that $f\in C^+[0,1]=\{f\in C[0,1]:f(x)\geq 0\}$.

Define a nonlinear operator H on the cone of positive continuous functions on $[0,1]:$

$$(Hf)(t)=\frac{\int_0^1\int_0^1 K(t,s,u)f(s)f(u)dsdu}{\int_0^1\int_0^1 K(0,s,u)f(s)f(u)dsdu}.$$

We'll study the existence of positive fixed points for the nonlinear operator H (i.e., solutions of the equation (9)). Put $C_0^+[0,1]=C^+[0,1]\setminus\{\theta\equiv 0\}$. Then the set $C^+[0,1]$ is the cone of positive continuous functions on $[0,1]$.

We define the Lyapunov integral operator $\mathcal{L}$ on $C[0,1]$ by the equality (see [3])

$$\mathcal{L}f(t)=\int_0^1 K(t,s,u)f(s)f(u)dsdu.$$

Put

$$\mathcal{M}_0=\left\{f\in C^+[0,1]:f(0)=1\right\}.$$

Lemma 3. *The equation $Hf=f$ has a nontrivial positive solution iff the Lyapunov equation $\mathcal{L}g=g$ has a nontrivial positive solution.*

Proof. At first we shall prove that the equation

$$Hf=f,\ \ f\in C_0^+[0,1] \tag{10}$$

has a positive solution iff the Lyapunov equation

$$\mathcal{L}g=\lambda g,\ \ g\in C^+[0,1] \tag{11}$$

has a positive solution in $\mathcal{M}_0$ for some $\lambda>0$.

Let λ_0 be a positive eigenvalue of the Lyapunov operator $\mathcal{L}$. Then there exists $f_0\in C_0^+[0,1]$ such that $\mathcal{L}f_0=\lambda_0 f_0$. Take $\lambda\in(0,+\infty)$, $\lambda\neq\lambda_0$. Define the function $h_0(t)\in C_0^+[0,1]$ by $h_0(t)=\frac{\lambda}{\lambda_0}f_0(t),\ \ t\in[0,1]$. Then $\mathcal{L}h_0=\lambda h_0$, i.e., the number λ is an eigenvalue of Lyapunov operator $\mathcal{L}$ corresponding the eigenfunction $h_0(t)$. It's easy to check that if the number $\lambda_0>0$ is an eigenvalue of the operator $\mathcal{L}$, then an arbitrary positive number is eigenvalue of the operator $\mathcal{L}$. Now we shall prove the lemma. Let equation (11) holds then the function $\frac{1}{\lambda}g(t)$ be a fixed point of the operator $\mathcal{L}$. Analogously, since H is non-linear operator we can correspond to the fixed point if there exist any eigenvector.

Proposition 4. *The equation*

$$\mathcal{L}f=\lambda f,\quad \lambda>0 \tag{12}$$

has at least one solution in $C_0^+[0,1]$.

Proof. Clearly, that the Lyapunov operator $\mathcal{L}$ is a compact on the cone $C^+[0,1]$. By the other hand we have

$$\mathcal{L}f(t)\geq m\left(\int_0^1 f(s)ds\right)^2,$$

for all $f\in C^+[0,1]$, where $m=\min K(t,s,u)>0$.

Put $\Gamma = \{f : \ \|f\| = r, \ \ f \in C[0,1]\}$. We define the set Γ_+ by

$$\Gamma_+ = \Gamma \cap C^+[0,1].$$

Then we obtain

$$\inf_{f \in \Gamma_+} \|\mathcal{L}f\| > 0.$$

Then by Schauder's theorem (see [4], p. 20) there exists a number $\lambda_0 > 0$ and a function $f_0 \in \Gamma_+$ such that, $\mathcal{L}f_0 = \lambda_0 f_0$.

Denote by $N_{fix.p}(H)$ and $N_{fix.p}(\mathcal{L})$ are the set of positive numbers of nontrivial positive fixed points of the operators $N_{fix.p}(H)$ and $N_{fix.p}(\mathcal{L})$, respectively. By Lemma 3 and Proposition 4 we can conclude that:

Proposition 5. *(a) The equation (10) has at least one solution in* $C_0^+[0,1]$.

(b) The equality $N_{fix.p}(H) = N_{fix.p}(\mathcal{L})$ *is hold.*

From Proposition 1 and the last Proposition we get the following theorem.

Theorem 6. *The set of splitting Gibbs measures corresponding to Hamiltonian (1) is non-empty.*

References

[1] Eshkabilov Yu. Kh, Haydarov F. H., Rozikov U. A. Uniqueness of Gibbs measure for models with uncountable set of spin values on a Cayley tree. Math. Phys. Anal. Geom., 2013, V.16, pp.1-17.

[2] Georgii H. O. Gibbs Measures and Phase Transitions, 2nd edn. De Gruyter Studies in Mathematics, Vol. 9. Walter de Gruyter, Berlin, 2011. – 545 p.

[3] Krasnosel'ski M. A. Positive Solutions of Opertor Equations. Fizmatgiz, Moscow, 1962 (Russian). – 394 p.

[4] Nirenberg L. Topics in nonlinear functional analysis. AMS, Courant Lec. Notes in Math, 6, N.Y., 2001. – 145 p.

[5] Rozikov U. A., Haydarov F. H. Four competing interactions for models with an uncountable set of spin values on a Cayley tree. Theor. Math. Phys., 2017, V.191, №3, pp.910-923.

[6] Rozikov U. A. Gibbs measures on Cayley trees. World Scientific, 2013. – 404 p.