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ON THE MANIFOLDS OF POSITIVE CURVATURE

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Abstract
In this paper invariant metrics on Lie group \( G = S^3 \times R \) are studied and it is found lower and upper bounds for the sectional curvature’s of the manifold \( G = S^3 \times R \).

Keywords: invariant metric, curvature, manifold, Lie group.
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1 Preliminaries

Let \( M \) be \( n \)-dimensional smooth Riemannian manifold with Riemannian metric \( g \) and \( V(M) \) be a set of smooth vector fields on \( M \). In article everywhere under the smoothness is understood the smoothness of class \( C^\infty \).

A Riemannian metric \( g \) induces a metric connection (Levi-Civita connection) \( \nabla \). Recall that, the connection is a mapping \( \nabla : V(M) \times V(M) \to V(M) \), denoted by \( (X,Y) \to \nabla_X Y \), and it has the following properties:

1. \( \nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \),
2. \( \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z \),
3. \( \nabla_{fX} Y = f\nabla_X Y \),
4. \( \nabla_X fY = X(f)Y + f\nabla_X Y \), where \( f \) is a smooth function.

Lie bracket of vector fields \( X, Y \) we denote by \( [X,Y] \). Levi-Civita connection \( \nabla \) and Lie bracket \( [X,Y] \) are related to

\( \nabla_X Y - \nabla_Y X = [X,Y] \).

Definition 1. The linear map \( R : V(M) \times V(M) \times V(M) \to V(M) \), denoted by the formula

\[ R_{XY} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \]

is called the curvature tensor and the scalar value \( k(X,Y) = \langle R_{XY} Y, X \rangle \) is called the Riemannian curvature [1].

Definition 2. The sectional curvature of the Riemannian manifold \( M \) in the point \( p \) for the two-dimensional direction \( \sigma \) determined by the vectors \( u, v \in T_pM \) is given by the rule

\[ K_{u,v} = \frac{k(u,v)}{|u|^2|v|^2 - \langle u,v \rangle^2} = \frac{\langle R_{uv} v, u \rangle}{|u|^2|v|^2 - \langle u,v \rangle^2}. \]
It is known that, if a manifold is a two-dimensional surface, immersed in a three-dimensional Euclidean space, then the sectional curvature coincides with the Gaussian curvature of the surface.

If the sectional curvature of $K_{u,v}$ is constant for all planes $\sigma$ in $T_pM$ and for every points $p \in M$, then the manifold $M$ is called a manifold of constant curvature.

Recall that a Lie group is a group $G$ that is in the time a smooth manifold, and its group actions and smooth structures are connected by the requirement that the maps $\varphi : G \to G$, $\psi : G \times G \to G$ defined by the equalities $\varphi(g) = g^{-1}$, $\psi(g,h) = gh$, maps $\varphi$ and $\psi$ were smooth.

The set of all real nonsingular $(n \times n)$ matrices with ordinary matrix multiplication forms a Lie group, denoted by $GL(n)$. Indeed, the set of all $(n \times n)$ matrices $A = (a^{ij})$, including singular ones, forms a vector space of dimension $n^2$. Its basis can serve as a matrix, one element of which is one, and the rest are zeros. Then $GL(n)$ is an open subset of $\mathbb{R}^{n^2}$ distinguished by the condition $\det A \neq 0$.

**Definition 3.** Each element $g$ of the Lie group $G$ can be associated with its two automorphisms $L_g : G \to G$ and $R_g : G \to G$ acting by the rule $L_g h = gh$, $R_g h = hg$. These maps are called left and right translations.

**Definition 4.** A vector field $X$ on the Lie group $G$ is called left-invariant if for any $g \in G$ the equality holds $dL_g X = X$.

For any vector $u \in T_eG$, where $e$ is the unit element of $G$, $X_g = (d_e L_g)u$ defines a left-invariant vector field.

The Riemannian metric $\langle \cdot, \cdot \rangle$ on a Lie group $G$ is called left-invariant, if for every $g, h \in G$ and for every tangent vectors $u, v \in T_gG$ it satisfies

$$\langle dL_h u, dL_h v \rangle_{hg} = \langle u, v \rangle_g.$$ (1)

The requirement (1) is equivalent to requiring that any left-invariant vector fields $\langle X, Y \rangle = \text{const}$. The Riemannian metric, which is both left-and right-invariant, is called bi-invariant. Such metrics do not exist on all Lie groups. The following Theorem holds [4].

**Theorem 1.** There exists at least one bi-invariant Riemannian metric on each compact Lie group.

Further, we give properties of bi-invariant metrics in the form of theorems [3].

**Theorem 2.** For a bi-invariant metric on the Lie group $G$ and any left-invariant vector fields $X, Y, Z$ on the Lie group $G$ the following equality is true

$$\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle.$$ 

**Theorem 3.** For left-invariant vector fields $X, Y$ the Levi-Civita connection of a bi-invariant Riemannian metric on the Lie group $G$ has the form

$$\nabla_X Y = \frac{1}{2} [X, Y].$$
Theorem 4. For a bi-invariant metric on the Lie group $G$, the sectional curvature at the point $p \in G$ in the direction $\sigma = X_p \wedge Y_p$ is expressed by the formula

$$K_\sigma (X_p \wedge Y_p) = \frac{1}{4} \left. \frac{\langle [X,Y], [X,Y] \rangle}{\langle X,X \rangle \langle Y,Y \rangle - \langle X,Y \rangle^2} \right|_p,$$

where $X,Y$ are left-invariant vector fields.

2 Main result

It is known that if $\varphi : M \to M$ is a differentiable mapping, then $[d\varphi X, d\varphi Y] = d\varphi [X,Y]$ [1].

Let the path $U(t) = (u^{ij}(t))$ exit from the unit $E$ of the group $GL(n)$. We denote its initial velocity by $X_E = (x^{ij})$, where $x^{ij} = \frac{d}{dt} u^{ij}(t) \mid_{t=0}$.

We define the field $X$ at the point $A = (a^{ij})$ as follows

$$X_A = dL_A(X_E) = \frac{d}{dt} \left( \sum_k a^{ik} u^{kj}(t) \right) = AX_E. \quad (*)$$

It follows from the relation (*), that the Lie bracket has the form $[X,Y] = XY - YX$.

We now consider the group $SO(n)$ of real orthogonal $(n \times n)$ matrices with the condition $\det A \neq 0$. The sectional curvature of the Lie group $SO(n)$ is expressed as follows [3]:

$$K_\sigma = \frac{1}{4} \frac{\sum_{i<j} \left( \sum_k X^{ik} y^{kj} - \sum_k y^{ik} X^{kj} \right)^2}{\sum_{i<j} (X^{ij})^2 \sum_{i<j} (y^{ij})^2 - \left( \sum_{i<j} X^{ij} y^{ij} \right)^2}.$$

In particular, for $SO(3)$, for example, $K_\sigma = \frac{1}{4}$.

We introduce a basis on the tangent space of the $T_E SO(3)$ as follows

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This basis is orthonormal and the Lie bracket of the basis elements is defined by the table

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

If we compare the basis vectors $\{1,0,0\}, \{0,1,0\}, \{0,0,1\}$ to the basis $X_1, X_2, X_3$ on $R^3$, then the linear isomorphism between $T_E SO(3)$ and $R^3$ is an isometry.

Consider the three-dimensional sphere $S^3$ of radius 2 with the standard Riemannian metric. The vectors $X_1, X_2, X_3$ in the standard metric of the sphere $S^3$ form an orthonormal basis in $T_e S^3$, and as left-invariant fields - an orthonormal basis for any
point $S^3$. The scalar product $\langle X, Y \rangle |_p = \lambda_1 \mu_1 + \lambda_2 \mu_2 + \lambda_3 \mu_3$, for any $X, Y \in T_p S^3$, such that

$$X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3,$$
$$Y = \mu_1 X_1 + \mu_2 X_2 + \mu_3 X_3.$$

We introduce a new metric $\langle \cdot, \cdot \rangle$ on $S^3$ as follows

$$\langle X, Y \rangle = \frac{1}{3} \lambda_1 \mu_1 + \lambda_2 \mu_2 + \lambda_3 \mu_3.$$

The difference between the metric $\langle \cdot, \cdot \rangle$ and the standard metric is that each tangent space “contracts” in the direction of the vector $X_1$.

Consider the Lie group $G = S^3 \times R$ with the metric of the direct product of sphere and line. This metric is bi-invariant. The Lie algebra of the group $G$ is obtained from the Lie algebra of the group $S^3$ by adding $X_1, X_2, X_3$ tangent to $R$ of the unit vector $X_4$ for which $[X_i, X_4] = 0$ holds for all $i$. The resulting basis is orthonormal.

**Theorem 5.** The manifold $G = S^3 \times R$ is a variety of strictly positive bounded sectional curvature.

**Proof.** Consider the vector fields on the group $G$

$$Z_1 = \frac{1}{\sqrt{3}} X_1 + \frac{\sqrt{2}}{\sqrt{3}} X_4, \quad Z_2 = -\frac{\sqrt{2}}{\sqrt{3}} X_1 + \frac{1}{\sqrt{3}} X_4.$$

Let $\pi : G \to S^3 \times \{0\}$ be a projection along the integral curves of the vector field $Z_2$. The sphere $S^3 \times \{0\}$ with the metric $\langle \cdot, \cdot \rangle$ is denoted by $M$. Moreover, the map $\pi : G \to M$ is a Riemannian submersion, where the field $Z_1$ is horizontal, and the field $Z_2$ is vertical, so that

$$|d\pi Z_1| = |Z_1| = 1, \quad |d\pi Z_2| = 0.$$

Since $X_1 = \frac{1}{\sqrt{3}} Z_1 - \sqrt{\frac{2}{3}} Z_2$, then $|d\pi X_1| = \frac{1}{\sqrt{3}}$.

Using the O’Neil’s formula, we calculate the sectional curvature $K_\sigma$ of the manifold $M$. Suppose that $X, Y \in T_p M$ is

$$X = \sqrt{3} \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \quad Y = \sqrt{3} \mu_1 X_1 + \mu_2 X_2 + \mu_3 X_3,$$

so that

$$|X \wedge Y|^2_M = \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2 = \nu_1^2 + \nu_2^2 + \nu_3^2,$$

where $\nu_1 = \lambda_2 \mu_3 - \lambda_3 \mu_2$, $\nu_2 = \lambda_3 \mu_1 - \lambda_1 \mu_3$, $\nu_3 = \lambda_1 \mu_2 - \lambda_2 \mu_1$.

Horizontal lifts of vector fields $X, Y$ in $G$ are vector fields

$$\overline{X} = \lambda_1 Z_1 + \lambda_2 X_2 + \lambda_3 X_3 = \frac{1}{\sqrt{3}} \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \sqrt{\frac{2}{3}} \lambda_1 X_4,$$
$$\overline{Y} = \mu_1 Z_1 + \lambda_2 X_2 + \lambda_3 X_3 = \frac{1}{\sqrt{3}} \mu_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \sqrt{\frac{2}{3}} \mu_1 X_4.$$
Their Lie bracket is
\[
[X,Y] = \nu_1 X_1 + \frac{1}{\sqrt{3}} \nu_2 X_2 + \frac{1}{\sqrt{3}} \nu_3 X_3
\]
and the vertical component is
\[
[X,Y]^V = -\sqrt{\frac{2}{3}} \nu_1 Z_2.
\]
The sectional curvature \(K_\sigma\) of the manifold \(M\) for \(\sigma = X \wedge Y\) is equal to
\[
K_\sigma = \frac{k(X,Y)}{|X \wedge Y|^2_M} = \frac{k(X,Y)_G + \frac{3}{4}([X,Y]^V)_G^2}{|X \wedge Y|^2_M} = \left(1 + 3 \left(\sqrt{\frac{2}{3}}\right)^2\right) \left(\frac{1}{4}\nu_1^2 + \frac{1}{\sqrt{3}} \left(\nu_2^2 + \nu_3^2\right)\right) = \frac{9}{12} \left(\nu_1^2 + (\nu_2^2 + \nu_3^2)\right).
\]
Thus, the sectional curvature of the manifold \(M\) is strictly positive and satisfies the inequality \(\frac{1}{12} \leq K_\sigma \leq \frac{3}{4}\). The theorem is proved. \(\square\)

References


