Scientific Bulletin of Namangan State University

Volume 1 | Issue 1 Article 3

5-10-2019

The uniqueness condition for a weakly periodic Gibbs measure for the hard-core model

Mukhtorjon Makhammadaliev
Namangan State University

The uniqueness condition for a weakly periodic Gibbs measure for the hard-core model

Cover Page Footnote

???????

Erratum

???????

UDK 517.98

HC MODELI UCHUN KUCHSIZ DAVRIY GIBBS O'LCHOVLARINING YAGONALIK SHARTLARI

Maxammadaliyev Muxtorjon Tursunmuxammad o'g'li
Namangan davlat universiteti, o'qituvchi

Annotatsiya: *Mazkur ishda Keli daraxtida HC modeli o`rganilgan. Bu model uchun normal bo`luvchining indeksi to`rt bo`lganda kuchsiz davriy Gibbs o'lchovlarining yagonalik shartlari topilgan.*

Kalit so'zlar: *Keli daraxti, konfiguratsiya, HC modeli, Gibbs o`lchovi, kuchsiz davriy Gibbs o'lchovi, translyatsion-invariant Gibbs o'lchovi.*

УСЛОВИЯ ЕДИНСТВЕННОСТИ СЛАБО ПЕРИОДИЧЕСКИХ МЕР ГИББСА ДЛЯ НС-МОДЕЛИ

Махаммадалиев Мухторжон Турсунмухаммад угли
Наманганский государственный университет, преподаватель

Аннотация: *Изучается НС модель на дереве Кэли. Для этой модели найдены условия единственности слабо периодических мер Гиббса в случае нормального делителя индекса четыре.*

Ключевые слова: *дерево Кэли, конфигурация, НС-модель, мера Гиббса, слабо периодическая мера Гиббса, трансляционно-инвариантная мера Гиббса.*

THE UNIQUENESS CONDITION FOR A WEAKLY PERIODIC GIBBS MEASURE FOR THE HARD-CORE MODEL

Makhammadaliev Mukhtorjon Tursunmukhammad ogli.
Namangan State University, teacher

Abstract: *We study the hard-core model on the Cayley tree. For this model we find conditions of uniqueness of weakly periodic Gibbs measures for a normal divisor of index four.*

Keywords: *Cayley tree, configuration, hard-core model, Gibbs measure, weakly periodic Gibbs measure, translation invariant Gibbs measure.*

Gibbs o`lchovi tushunchasi statistik mexanikada muhim rol o'ynaydi. Berilgan gamiltonianning asosiy vazifasi bu gamiltonianga mos barcha Gibbs o'lchovlarini tasniflashdan iborat. Gibbs o'lchovining ta'rifi va Gibbs o'lchovlari nazariyasiga oid boshqa tushunchalar bilan [1]-[4] ishlarda tanishish mumkin. Keli daraxtida Izing modeli uchun bu masala yetarlicha ko'p o'rganilgan.

τ^k - bu $k \geq 1$ tartibli cheksiz Keli daraxti, ya`ni siklsiz graf bo'lib, har bir uchidan $k+1$ ta qirra chiqadi. Faraz qilaylik, $\tau^k = (V, L, i)$ bo'lsin, bu yerda $V - \tau^k$ ning uchlari to'plami, $L-$uning qirralari to'plami va $i-$insidentlik funksiyasi bo'lib, har bir $l \in L$ qirraga uning chetki uchlari bo`lgan $x, y \in V$ larni mos qo'yadi. Agar $i(l) = \{x, y\}$ bo'lsa, u holda x va y lar eng yaqin qo'shnilar deyiladi va $l = \langle x, y \rangle$

kabi belgilanadi. Keli daraxtida $d(x,y)$ $(x,y\in V)$ masofa ushbu formula bilan aniqlanadi:

$$d(x,y)=\min\{d\,|\,\exists x=x_0,x_1,\ \dots,\ x_{d-1},x_d=Y:\ \langle x_0,x_1\rangle,\dots,\langle x_{d-1},x_d\rangle\}.$$

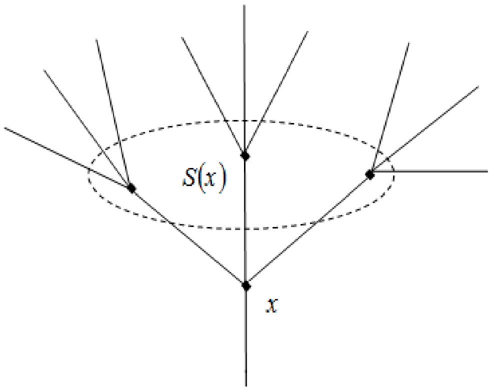
Fiksirlangan $x^0\in V$ uchun ushbu belgilashlar kiritiladi:

$$W_n=\{x\in V\ |\ d(x,x^0)=n\},\quad V_n=\{x\in V\ |\ d(x,x^0)\le n\},$$

$x\in W_n$ uchun esa quyidagicha belgilash kiritamiz:

$$S(x)=\{y\in W_{n+1}:\ d(x,y)=1\}.$$

$S(x)$ to`plam $x$ uchning to`g`ri avlodlari deyiladi (1-rasm).



1-rasm. x uchning to`g`ri avlodlari.

$\Phi=\{0,1\}$ va $\sigma\in\Phi^V$ – konfiguratsiya bo`lsin, ya`ni $\sigma=\{\sigma(x)\in\Phi: x\in V\}$, bu yerda $\sigma(x)=1$ sharti Keli daraxtida x uch "band" ekanligini, $\sigma(x)=0$ sharti esa x uch "bo'sh" ekanligini bildiradi.

Ta`rif-1. Agar V (mos ravishda V_n yoki W_n) dagi har qanday $\langle x,y\rangle$ qo'shnilar uchun $\sigma(x)\sigma(y)=0$ bo'lsa, u holda σ konfiguratsiya joiz konfiguratsiya deyiladi va bunday konfiguratsiyalar to'plamini Ω (Ω_{V_n} va Ω_{W_n}) kabi belgilanadi. Ravshanki, $\Omega\subset\Phi^V$.

HC modelining gamiltoniani quyidagi formula orqali aniqlanadi:

$$H(\sigma)=\begin{cases} J\sum\limits_{x\in V}\sigma(x), & \text{agar } \sigma\in\Omega,\\ +\infty, & \text{agar } \sigma\notin\Omega,\end{cases}$$

bunda $J\in R$.

$\sigma_n\in\Omega_{V_n}$ joiz konfiguratsiya uchun belgilash kiritamiz:

$$\#\sigma_n=\sum_{x\in V_n}\sigma_n(x),$$

$\#\sigma_n$ – bu V_n dagi birlar (band uchlar) soni.

$z: x\mapsto z_x=(z_{0,x},z_{1,x})\in R_+^2$ vektor-funksiya V da berilgan bo'lsin. U holda $\lambda>0$ va $n=1,2,\dots$ lar uchun Ω_{V_n} da quyidagicha aniqlangan $\mu^{(n)}$ ehtimollik o'lchovini qaraylik:

$$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n}\lambda^{\#\sigma_n}\prod_{x\in W_n} z_{\sigma_n(x),x}. \qquad (1)$$

Bu yerda Z_n –normallovchi bo'luvchi:

$$Z_n = \sum_{\varphi_n\in\Omega_{V_n}} \lambda^{\#\varphi_n}\prod_{x\in W_n} z_{\varphi_n(x),x}.$$

$\mu^{(n)}$ ehtimollik o'lchovlari ketma-ketligi muvofiqlik shartini qanoatlantiradi deyiladi, agar ixtiyoriy $n\geq 1$ va $\sigma_{n-1}\in\Omega_{V_{n-1}}$ lar uchun quyidagi tenglik o'rinli bo'lsa:

$$\sum_{\omega_n\in\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1}\vee\omega_n)\mathbf{1}(\sigma_{n-1}\vee\omega_n\in\Omega_{V_n}) = \mu^{(n-1)}(\sigma_{n-1}), \qquad (2)$$

bunda

$$\mathbf{1}(\sigma_{n-1}\vee\omega_n\in\Omega_{V_n}) = \begin{cases} 1, \text{ agar } & \sigma_{n-1}\vee\omega_n\in\Omega_{V_n} \\ 0, \text{ agar } & \sigma_{n-1}\vee\omega_n\notin\Omega_{V_n}. \end{cases}$$

Bunday holda Kolmogorov teoremasiga ko\`ra $(\Omega,\mathbf{B})$ da shunday yagona μ o'lchov mavjudki, $\forall n\in N$ va $\sigma_n\in\Omega_{V_n}$ lar uchun

$$\mu(\{\sigma\in\Omega:\ \sigma|_{V_n}=\sigma_n\}) = \mu^{(n)}(\sigma_n)$$

tenglik o'rinli bo'ladi. Bu yerda $\mathbf{B}-\Omega$ ning silindrik qism to\`plamlaridan hosil bo\`luvchi σ-algebradir.

Ta'rif-2. (1) formula orqali aniqlangan (2) muvofiqlik shartini qanoatlantiruvchi μ o'lchov $\lambda>0$ bilan $z: x\in V\setminus\{x_0\}\mapsto z_x$ funksiyaga mos keluvchi HC-Gibbs o'lchovi deyiladi.

[5] dan ma\`lumki, Keli daraxtida HC modelining har bir Gibbs o'lchoviga

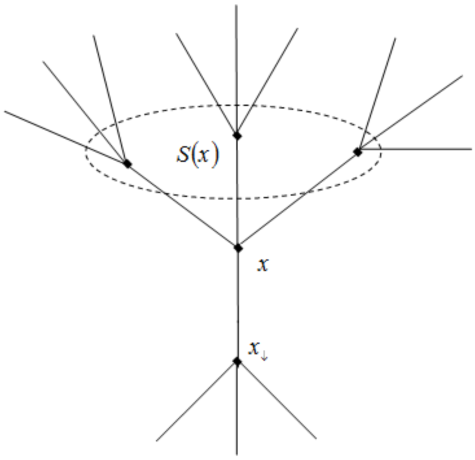
$$z'_x = \prod_{y\in S(x)}\frac{1}{1+\lambda z'_y}$$

tenglamani qanoatlantiruvchi $z=\{z_x,\ x\in G_k\}$ miqdorlar to'plamini mos qo'yish mumkin, bu yerda $z'_x = \dfrac{z_{1,x}}{z_{0,x}}$, $\lambda = e^{J\beta}>0$ –parametr va $\beta=\dfrac{1}{T}, T>0$.

Faraz qilaylik, $G_k^* - G_k$ ning qism gruppasi bo'lsin.

Ta\`rif-3. Agar har qanday $x\in G_k$, $y\in G_k^*$ lar uchun $z_{yx}=z_x$ bo'lsa, u holda $z=\{z_x,\ x\in G_k\}$ miqdorlar G_k^*-davriy deyiladi. G_k-davriy miqdorlar translyatsion-invariant deyiladi.

Har qanday $x\in G_k$ uchun $\{y\in G_k:\langle x,y\rangle\}\setminus S(x)$ to'plam yagona elementdan iborat va uni $x_\downarrow$ deb belgilaymiz. Uni, ya\`ni $x_\downarrow$ ni x uchning ajdodi deyiladi (2-rasm).



2-rasm. x **uchning ajdodi.**

Faraz qilaylik, $G_k / G_k^* = \{H_1, ..., H_r\}$ faktor-gruppa bo`lsin, bu yerda $G_k^* - r \geq 1$ indeksli normal bo'luvchi.

Ta`rif-4. Agar $x \in H_i$, $x_\downarrow \in H_j$, $\forall x \in G_k$ lar uchun $z_x = z_{ij}$ bo'lsa, $z = \{z_x, x \in G_k\}$ miqdorlar G_k^*-kuchsiz davriy deyiladi.

Eslatma. Agar z_x ning qiymati $x_\downarrow$ ga bog'liq bo'lmasa, u holda kuchsiz davriy o'lchovlar davriy o'lchovlar bilan ustma- ust tushadi.

Ta`rif-5. Agar μ o'lchov G_k^*-(kuchsiz) davriy z miqdorlarga mos kelsa, u holda μ o'lchov G_k^*-(kuchsiz) davriy deyiladi.

Mazkur ishda biz to`rt indeksli $G_k^{(4)}$ normal bo'luvchi uchun $G_k^*$-kuchsiz davriy Gibbs o'lchovlarini o`rganamiz.

$A \subset \{1, 2, ..., k+1\}$ va $H_A = \{x \in G_k : \sum_{i \in A} w_x(a_i) - \text{juft son}\}$ bo`lsin, bunda $w_x(a_i) - \text{son}$ $x \in G_k$ so`zdagi a_i harflar soni, $G_k^{(2)} = \{x \in G_k : |x| - \text{juft son}\}$, $|x| -$ bu $x \in G_k$ so`z uzunligi va $G_k^{(4)} = H_A \cap G_k^{(2)} -$ indeksi to`rtga teng bo'lgan normal bo'luvchi.

$G_k / G_k^{(4)} = \{H_0, H_1, H_2, H_3\}$ faktor gruppani qaraymiz, bunda

$$H_0 = \left\{ x \in G_k : |x| - \text{juft}, \sum_{j \in A} w_j(x) - \text{juft} \right\},$$

$$H_1 = \left\{ x \in G_k : |x| - \text{juft}, \sum_{j \in A} w_j(x) - \text{toq} \right\},$$

$$H_2 = \left\{ x \in G_k : |x| - \text{toq}, \sum_{j \in A} w_j(x) - \text{juft} \right\},$$

$$H_3 = \left\{ x \in G_k : |x| - \text{toq}, \sum_{j \in A} w_j(x) - \text{toq} \right\}.$$

U holda $G_k^{(4)}$-kuchsiz davriy z_x miqdorlar quyidagi ko'rinishga ega:

$$z_x = \begin{cases} z_1, & x \in H_3, x_\downarrow \in H_1 \\ z_2, & x \in H_1, x_\downarrow \in H_3 \\ z_3, & x \in H_3, x_\downarrow \in H_0 \\ z_4, & x \in H_0, x_\downarrow \in H_3 \\ z_5, & x \in H_1, x_\downarrow \in H_2 \\ z_6, & x \in H_2, x_\downarrow \in H_1 \\ z_7, & x \in H_2, x_\downarrow \in H_0 \\ z_8, & x \in H_0, x_\downarrow \in H_2. \end{cases}$$

Bunda z_x lar quyidagi tenglamalar sistemasini qanoatlantiradi:

$$\begin{cases} z_1 = \dfrac{1}{(1+\lambda z_4)^i} \cdot \dfrac{1}{(1+\lambda z_2)^{k-i}}, & z_2 = \dfrac{1}{(1+\lambda z_6)^i} \cdot \dfrac{1}{(1+\lambda z_1)^{k-i}}, \\ z_3 = \dfrac{1}{(1+\lambda z_4)^{i-1}} \cdot \dfrac{1}{(1+\lambda z_2)^{k-i+1}}, & z_4 = \dfrac{1}{(1+\lambda z_3)^{i-1}} \cdot \dfrac{1}{(1+\lambda z_7)^{k-i+1}}, \\ z_5 = \dfrac{1}{(1+\lambda z_6)^{i-1}} \cdot \dfrac{1}{(1+\lambda z_1)^{k-i+1}}, & z_6 = \dfrac{1}{(1+\lambda z_5)^{i-1}} \cdot \dfrac{1}{(1+\lambda z_8)^{k-i+1}}, \\ z_7 = \dfrac{1}{(1+\lambda z_5)^i} \cdot \dfrac{1}{(1+\lambda z_8)^{k-i}}, & z_8 = \dfrac{1}{(1+\lambda z_3)^i} \cdot \dfrac{1}{(1+\lambda z_7)^{k-i}}. \end{cases} \quad (3)$$

[6] ishdan ma'lumki, ushbu

$I_1 = \{(z_1, z_2, z_7, z_8) \in R^4 : z_1 = z_2 = z_7 = z_8\}$, $I_2 = \{(z_1, z_2, z_7, z_8) \in R^4 : z_1 = z_7, z_2 = z_8\}$,
$I_3 = \{(z_1, z_2, z_7, z_8) \in R^4 : z_1 = z_2, z_7 = z_8\}$, $I_4 = \{(z_1, z_2, z_7, z_8) \in R^4 : z_1 = z_8, z_2 = z_7\}$.
to\`plamlar (3) akslantirishga nisbatan invariantdir.

Yana [6] va [7] ishlardan quyidagi teorema ma'lum.

Теорема 1. [6],[7] HC modeli uchun normal bo\`luvchining indeksi to\`rt bo\`lganda quyidagi tasdiqlar o\`rinli:

1. $k \geq 1$ va $i \leq k$ bo\`lsin. U holda $I_1$ da kuchsiz davriy Gibbs o'lchovi yagona. Bundan tashqari bu o\`lchov yagona translyatsion invariant Gibbs o\`lchovi bilan ustma-ust tushadi.

2. $k = 2$, $\lambda_{cr} = 4$ va $i = 1$ yoki $i = 2$ bo\`lsin. U holda $I_2$ da $\lambda < \lambda_{cr}$ bo\`lganda yagona, $\lambda = \lambda_{cr}$ da ikkita va $\lambda > \lambda_{cr}$ da uchta kuchsiz davriy Gibbs o'lchovlari mavjud, bunda har bir holda Gibbs o\`lchovlaridan bittasi translyatsion invariant, qolganlari davriy bo\`lmagan kuchsiz davriy Gibbs o\`lchovlaridir.

3. $k = 3, i = 1$ bo\`lsin. U holda shunday $\lambda_0$ mavjudki, $I_2$ da $\lambda > \lambda_0$ bo\`lganda kamida to\`rtta Gibbs o\`lchovlari mavjud, bunda ulardan bittasi bittasi translyatsion-invariant, qolganlari davriy bo\`lmagan kuchsiz davriy Gibbs o\`lchovlaridir.

4. $k \geq 1, i = 1$ yoki $k = 2, i = 2$ bo\`lsin. U holda I_3 da kuchsiz davriy Gibbs o'lchovi yagona.

5. $k=2,3$ va $i=1$ yoki $k=i$ bo`lsin. U holda I_4 da kuchsiz davriy Gibbs o'lchovi yagona.

Quyidagi teorema o`rinli.

Teorema 2. $I_3=\{(z_1,z_2,z_7,z_8)\in R^4: z_1=z_2, z_7=z_8\}$ invariantda $k=3, i=3$ va $k=4, i=4$ bo'lganda HC-modeli uchun barcha kuchsiz davriy Gibbs o'lchovlari translyatsion invariant bo'ladi.

Isbot. Teoremani isbotlash uchun $\lambda>0$ bo'lganda (3) tenglamalar sistemasi faqat $z_1=z_2=z_7=z_8$ ko'rinishdagi yechimga ega ekanligini ko'rsatish yetarli.

I_3 invariantda $k=3, i=3$ bo'lganda (3) tenglamalar sistemasi quyidagi ko'rinishda bo'ladi:

$$\begin{cases} z_1=\left(\dfrac{1+\lambda z_7}{1+\lambda z_7+\lambda\sqrt[3]{z_7^2}}\right)^3 \\ z_7=\left(\dfrac{1+\lambda z_1}{1+\lambda z_1+\lambda\sqrt[3]{z_1^2}}\right)^3 \end{cases} \quad (4)$$

(4) dan ko'rish mumkinki $z_1>0$ va $z_7>0$ bo'lganda $z_1<1$ va $z_7<1$ bo'ladi, ya'ni biz uchun musbat yechimlar zarur. Quyidagicha belgilash kiritamiz: $y^3=z_1$, $x^3=z_7$. U holda $0<x<1, 0<y<1$ bo'lib quyidagiga ega bo'lamiz:

$$\begin{cases} y=\dfrac{1+\lambda x^3}{1+\lambda x^3+\lambda x^2} \\ x=\dfrac{1+\lambda y^3}{1+\lambda y^3+\lambda y^2}. \end{cases} \quad (5)$$

Maxrajni nolga teng emasligini hisobga olib, uni chap tomonga ko'paytiramiz:

$$\begin{cases} x+\lambda xy^2+\lambda xy^3=1+\lambda y^3 \\ y+\lambda x^2 y+\lambda x^3 y=1+\lambda x^3. \end{cases} \quad (6)$$

(6) ning birinchi tenglamasidan ikkinchisini ayiramiz va ko'paytuvchilarga ajratamiz:

$$(x-y)-\lambda xy(x-y)-\lambda xy(x^2-y^2)=\lambda(y^3-x^3). \quad (7)$$

(7) tenglamani yechimlari mos ravishda (5) ning ham yechimlari bo'ladi.

Agar $x=y$ bo'lgan holda $z_1=z_7$ bo'lib, yechim I_1 invariantga tushib qoladi. I_1 invariantda esa yechim yagonaligi isbotlangan.

Endi $x\neq y$ bo`lsin. U holda $x<y$ yoki $x>y$ lardan faqat bittasi o'rinli bo'ladi. (7) ning ikki tomonini $x-y$ ga bo`lamiz:

$$1-\lambda xy-\lambda xy(x+y)=-\lambda(x^2+xy+y^2).$$

Oxirgi tenglikni soddalashtirganimizdan so'ng ushbu

$$\lambda(x^2y+xy^2-x^2-y^2)=1$$

ko'rinishiga keladi. Bundan

$$\lambda = \frac{1}{x^2(y-1)+y^2(x-1)}$$

tenglikni hosil qilamiz. Ma'lumki, $0<x<1, 0<y<1$. Bundan $\lambda<0$ ga ega bo`lamiz, lekin $\lambda>0$ edi. Demak, (7) tenglama $x \neq y$ da yechimga ega emas ekan.

I_3 invariantda $k=4, i=4$ bo'lganda (3) tenglamalar sistemasi quyidagi ko'rinishda bo'ladi:

$$\begin{cases} z_1 = \left(\dfrac{1+\lambda z_7}{1+\lambda z_7+\lambda\sqrt[4]{z_7^3}}\right)^4 \\ z_7 = \left(\dfrac{1+\lambda z_1}{1+\lambda z_1+\lambda\sqrt[4]{z_1^3}}\right)^4 \end{cases} \quad (8)$$

(8) dan ko'rish mumkinki, $z_1>0$ va $z_7>0$ bo'lganda $z_1<1$ va $z_7<1$ bo'ladi. Belgilash kiritamiz: $y^4=z_1$, $x^4=z_7$. U holda $0<x<1,\ 0<y<1$ bo'lib, quyidagiga ega bo'lamiz:

$$\begin{cases} y = \dfrac{1+\lambda x^4}{1+\lambda x^4+\lambda x^3} \\ x = \dfrac{1+\lambda y^4}{1+\lambda y^4+\lambda y^3}. \end{cases} \quad (9)$$

Maxrajni nolga teng emasligini hisobga olib, uni chap tomonga ko'paytiramiz:

$$\begin{cases} x+\lambda xy^3+\lambda xy^4 = 1+\lambda y^4 \\ y+\lambda x^3y+\lambda x^4y = 1+\lambda x^4 \end{cases} \quad (10)$$

(10) ning birinchi tenglamasidan ikkinchisini ayiramiz va ko'paytuvchilarga ajratamiz:

$$(x-y)-\lambda xy(x^2-y^2)-\lambda xy(x^3-y^3)=\lambda(y^4-x^4) \quad (11)$$

(11) tenglamani yechimlari mos ravishda (9) ning ham yechimlari bo'ladi. Yuqoridagi kabi, agar $x=y$ bo`lsa, yechim yagona bo`ladi.

$x\neq y$ bo`lsin. U holda $x<y$ yoki $x>y$ lardan faqat bittasi o'rinli bo'ladi. Faraz qilaylik $x>y$ bo'lsin. U holda (11) ni har ikki tomonini $y-x$ ga bo'lib yuboramiz:

$$-1+\lambda xy(x+y)+\lambda xy(x^2+xy+y^2)=\lambda(x^2+y^2)(x+y).$$

Bu tenglamani yechimi yo`qligini isbotlash uchun

$$\lambda xy(x+y)+\lambda xy(x^2+xy+y^2)<\lambda(x^2+y^2)(x+y)$$

ekanligini isbotlash yetarli. Ma'lumki, $0<x<1,\ 0<y<1$. Bundan

$$\lambda xy+\lambda xy(x+y)<\lambda(x^2+xy+y^2).$$

Tengsizlikni har ikki tomonini $x>0$ ga ko`paytiramiz. U holda

$$\lambda x^2y+\lambda xy(x^2+xy)<\lambda(x^3+x^2y+xy^2)$$

ga ega bo`lamiz. Natijaning har ikkala tomoniga $\lambda xy+(y+y^2)+\lambda y^3$ ni qo`shamiz:

$$\lambda xy(x+y)+\lambda xy(x^2+xy+y^2)+\lambda y^3<\lambda(x^2+y^2)(x+y)+\lambda xy(y+y^2).$$

$\lambda y^3<\lambda xy(y+y^2)$ ekanligini inobatga olib, har ikki tengsizlikni bir-biridan ayirsak

$$\lambda xy(x+y)+\lambda xy(x^2+xy+y^2)<\lambda(x^2+y^2)(x+y)$$

hosil bo'ladi. Bu esa yuqoridagi (9) sistema faqatgina $x=y$ bo'lgandagina yechimga ega ekanligini bildiradi. Xuddi shu jarayonni $x<y$ uchun ham bajariladi. Unda faqat tengsizlikni $y>0$ ga ko`paytiramiz. Bu esa o'z navbatida $I_3$ invariantda $k=3,\ i=3$ va $k=4, i=4$ bo'lganda kuchsiz davriy Gibbs o'lchovlari translyatsion invariant bo'lishini ko'rsatadi. Teorema isbot bo`ldi.

References:

1. Georgi X.O. Gibbsovskie meri i fazovie perexodi. - M.: Mir, 1992.
2. Preston K. Gibbsovskie sostoyaniya na schetnix mnojestvax. - M. : Mir, 1977.
3. Sinay Ya. G. Teoriya fazovix perexodov. Strogie rezultati. - M.: Nauka, 1980.
4. Rozikov U. A. Gibbs measures on Cayley trees. Singapore: World Sci., 2013.
5. Suhov Yu. M., Rozikov U. A. A hard-core model on a Cayley tree: an example of a loss network. Queueing Syst.. 2004. V. 46. P. 197–212.
6. Xakimov R. M. Slabo periodicheskie meri Gibbsa dlya NS-modeli dlya normalnogo delitelya indeksa chetire. Ukr. mat. jurn.. 2015. T. 67, № 10. S. 1409–1422.
7. Khakimov R.M., Madgoziyev G.T. Weakly periodic Gibbs measures for two and three state HC models on a Cayley tree. UzMJ, 2018, -№ 3, p. 116-131.