ACTA TTPU

I. FUNDAMENTAL SCIENCE

QUASI-SYMMETRIC DISTRIBUTION FUNCTION OF INVARIANT MEASURE OF CIRCLE HOMEOMORPHISMS WITH SINGULARITIES

U.A. Safarov

Turin polytechnic university in Tashkent
safarovua@mail.ru

Abstract

Let f beacirclehomeomorphismwithsinglecriticalpointofnon-integerorder,thatis, $f(x)=(x-x_{cr})|x-x_{cr}|^{d-1}+f(x_{cr})$

, $d>2$, for some δ-neighborhood $U_{\delta}(x_{cr})$. We prove that, if the homeomorphism f is P-homeomorphism on the set $S^1 \setminus U_{\delta}(x_{cr})$ with irrational rotation number ρ_f, then f is topologically conjugate to the pure rotation f_ρ.

Moreover, φ is quasi-symmetric if and only if ρ_f is of bounded type.

Key words: Circle homeomorphism, rotation number, critical point, break point, invariant measure.

In this work we study the some properties of distribution function of invariant measure of critical circle maps with non-integer order and with several break points.

Let f be an orientation preserving homeomorphism of the circle $S^1 \simeq R^1/Z^1$ with lift $F: R^1 \to R^1$, which is continuous, strictly increasing and fulfills $F(x+1)=x+1,\ \ x\in R^1$. The most important arithmetic characteristic of the homeomorphism f of the unit circle S^1 is the rotation number:

$$\rho_f=\lim_{n\to\infty}\frac{F^n(x)}{n}\ (\mathrm{mod}\,1),\ \ x\in R^1.$$

Henceforth, F^n denotes the n th iterate of the function F. The rotation number is rational if and only if f has periodic orbits. Denjoy proved that if f is a circle diffeomorphism with irrational rotation number $\rho=\rho_f$ and $\log f'$ is of bounded variation, then f is topologically conjugate to the pure rotation $f_\rho : x\to x+\rho(\mathrm{mod}\,1)$; that is, there exists an essentially unique homeomorphism φ of the circle with $\varphi\circ f=f_\rho\circ\varphi$ (see [1]). Since the conjugating map φ and the unique f-invariant measure μ_f are related by $\varphi(x)=\mu_f([0;x]),\ x\in S^1$

(see [1]), regularity properties of the conjugating map φ imply corresponding properties of the density of the absolutely continuous invariant measure μ_f as a distribution function on the circle. The problem of relating the smoothness of φ to that of f has been studied extensively. Indepth results have been found; see [2–5].

Other classes of circle homeomorphisms are critical circle homeomorphisms and circle diffeomorphisms with several break points.

I. Critical Circle Homeomorphisms. The orientation preserving circle homeomorphisms f, such that $f \in C^r$, $r \geq 3$, have a critical point x_{cr}, around which, in some C^r coordinate system, f has the form $f(x) = \phi(x)\,|\,\phi(x)|^{d-1} + f(x_{cr})$ for all $x \in U_\delta(x_{cr})$,

where $\phi : U_\delta(x_{cr}) \to \phi(U_\delta(x_{cr}))$ is a C^r diffeomorphism such that $\phi(x_{\tilde{n}r}) = 0$, and $d > 1$.

Such critical point is called non-flat critical point of order d.

An important one-parameter family of examples of critical circle maps are the

Arnold's maps defined by

$$f_\theta(x) = x + \theta + \frac{1}{2\pi}\sin 2\pi x (\mathrm{mod} 1),\ x \in S^1$$

For every $\theta \in R^1$ the map f_θ is a critical map with critical point 0 of cubic type.

II. P-Homeomorphisms. That is, orientation preserving circle homeomorphisms f are differentiable except in many countable points called break points admitting left and right derivatives (denoted by f'_- and f'_-, resp.) such that

(i) there exist some constants $0 < a < b < \infty$ such that

$a < f'(x) < b$ for all $x \in S^1 \setminus BP(f)$ and

$a < f'_-(x_b), f'_+(x_b) < b$ for all $x_b \in BP(f)$,

where $BP(f)$ denotes the set of the break points of f;

(ii) $\log f'$ has bounded variation: $v = \operatorname*{var}_{S^1} \log f' < \infty$.

The ratio $\sigma_f(x_b) = \dfrac{f'_-(x_b)}{f'_+(x_b)}$ is called jump ratio of f at x_b.

The existence of the conjugating map for the class critical circle homeomorphisms was proved by Yoccoz in [7] and for the class P-homeomorphisms the existence of conjugating map was proved by Herman in [2].

The singularity of the conjugating map for critical circle homeomorphisms was shown by Graczyk and Swiątek in [8]. T hey proved that if f is C^3 smooth circle homeomorphism with infinitely many critical points of polynomial type and an irrational rotation number of bounded type, then the conjugating map φ is a singular function. For the P-homeomorphisms, the situation is different; that is, in this case, the conjugating map can be singular or absolutely continuous. Indeed, in the works [9–11], it was shown that the conjugating map is singular. The deeper result in this area was obtained by Dzhalilov et al. [12]. They proved that if f is piecewise-smooth P-homeomorphism with infinite number of break points and the product of jump ratios at these break points is nontrivial, then the conjugating map is a singular function. But in the works [9, 13], it was shown that if f is piecewise-smooth P-homeomorphism with infinite number of break points having the (D)-property (see for the definition [13]) and the product of the jump ratios on each orbit is equal to 1, then the conjugating map is an absolutely continuous function. Now, we discuss the symmetric property of a given function.

Definition 1. A homeomorphism f is called quasi-symmetric if there exists a constant $K > 0$ such that for any $x \in S^1$ and $\delta \neq 0$ the following inequality holds:

$$\frac{|f(x+\delta) - f(x)|}{|f(x) - f(x-\delta)|} < K.$$

The criteria of quasi-symmetry of the conjugating map of the critical circle homeomorphisms were obtained by Swiątek in [14]. Due to [14], if the circle homeomorphism with an irrational rotation number is analytic and has infinitely many critical points, then the conjugating map is quasi-symmetric if and only if the rotation number is of bounded type.

The quasi-symmetric property of the conjugating map of

P-homeomorphisms is also different from the case of critical circle homeomorphisms. More precisely, if the rotation number of P-homeomorphism is irrational of bounded type, then conjugating map is quasi-symmetric, but there is a P-homeomorphism with irrational rotation number of unbounded type such that the conjugating map is quasi-symmetric. Now, we introduce our class.

Let f be a circle homeomorphism.

(a) f has one critical point polynomial type of order $d > 2$ and

$$f(x) = (x - x_{cr})\,|\,x - x_{cr}\,|^{d-1} + f(x_{cr})$$ for

some δ-neighborhood $U_\delta(x_{cr})$;

(b) f is a P-homeomorphism on the set $S^1 \setminus U_\delta(x_{cr})$.

Now, we state our main results.

Theorem 1. Suppose that a circle homeomorphism f satisfies the conditions (a)–(b) and the rotation number ρ_f is irrational. Then, there exists circle homeomorphism $\varphi : S^1 \to S^1$,

such that the functional equation

$$\varphi(x + \rho_f) = f(\varphi(x)),\ x \in S^1$$

is satisfied. Moreover, φ is quasi-symmetric if and only if ρ_f is of bounded type.

Note that the result of Theorem 1 was obtained by Dzhalilov, Noorani and Akhatkulov [15] for critical circle homeomorphisms with odd order of critical point. In our case the order of critical point can be any real number bigger than 2. The present paper is a continuation of [15] and in a certain sense complements the results obtained in that paper.

References

1. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, Ergodic theory, vol. 245, Springer, Berlin, Germany, 1982.

2. M. Herman, "Sur la conjugaison diffe'rentiable des diffe'omorphismes du cercle a' des rotations", Institut des Hautes E'tudes Scientiiques, vol. 49, pp. 5–234, 1979.

3. Y. Katznelson and D. Ornstein, "The differentiability of the conjugation of certain diffeomorphisms of the circle," Ergodic theory and Dynamical Systems, vol. 9, no.4, pp.643–680, 1989.

4. Y. Katznelson and D. Ornstein, "The absolute continuity of the conjugation of certain diffeomorphisms of the circle," Ergodic theory and Dynamical Systems, vol. 9, no.4, pp.681–690, 1989.

5. K. M. Khanin and Ya. G. Sinai, "Smoothness of conjugacies of diffeomorphisms of the circle with rotations," Russian Mathematical Surveys, vol.44, no.1, pp.69–99, 1989, translation of Uspekhi Matematicheskikh Nauk,vol.44, pp.57–82, 1989.

6. W. de Melo and S. van Strien, One-Dimensional Dynamics, vol. 25, Springer, Berlin, Germany, 1993.

7. J.-C. Yoccoz, "Il n'y a pas de contre-exemple de Denjoy analytique," Comptes Rendus des Se'ances de l'Acade'mie des Sciences, vol. 298, no. 7, pp. 141–144, 1984.

8. J. Graczyk and G. Swiątek, "Singular measures in circle dynamics," Communications in Mathematical Physics, vol.157, no.2, pp.213–230, 1993.

9. Kh. Akhadkulov, "Some circle homeomorphisms with break type singularities," Russian Mathematical Surveys,vol.61,no.5, pp.981–983, 2006.

10. A. A. Dzhalilov and I. Liousse, "Circle homeomorphisms with two break points," Nonlinearity,vol.19,no.8, pp.1951–1968, 2006.

11. A. A. Dzhalilov, I. Liousse, and D. Mayer, "Singular measures of piecewise smooth circle homeomorphisms with two break points. "Discrete and Continuous Dynamical Systems, vol.24, no. 2, pp. 381–403, 2009.

12. A. A.Dzhalilov, D.Mayer, and U. A. Safarov, "Piecewise-smooth circle homeomorphisms with several break points," Izvestiya, vol. 76, no. 1, pp. 94–112, 2012.

13. A. Adouani and H. Marzougui, "Singular measures for class P-circle homeomorphisms with several break points," Ergodic theory and Dynamical Systems, pp.1–34, 2012.

14. G. Swiątek, "On critical circle homeomorphisms," Boletim da Sociedade Brasileira de Matematica,vol.29, no.2,pp.329–351, 1998.

15. A. Dzhalilov, S. Noorani and S. Akhatkulov, "On Critical Circle Homeomorphisms with Infinite Number of Break Points". Abstract and Applied Analysis Volume 2014, Article ID 378742, 6 pages, http://dx.doi.org/10.1155/2014/378742.