QUASI-SYMMETRIC DISTRIBUTION FUNCTION OF INVARIANT MEASURE OF CIRCLE HOMEOMORPHISMS WITH SINGULARITIES

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Abstract
Let $f$ be a circle homeomorphism with a single critical point of non-integer order, that is,
$$f(x) = (x - x_{cr}) |x - x_{cr}|^{d-1} + f(x_{cr})$$
$\delta > 2$, for some $\delta$-neighborhood $U_{\delta}(x_{cr})$. We prove that, if the homeomorphism $f$ is $P$-homeomorphism on the set $S^1 \setminus U_{\delta}(x_{cr})$ with irrational rotation number $\rho_f$, then $f$ is topologically conjugate to the pure rotation $f_\rho$.

Moreover, $\varphi$ is quasi-symmetric if and only if $\rho_f$ is of bounded type.

Key words: Circle homeomorphism, rotation number, critical point, break point, invariant measure.

In this work we study the some properties of distribution function of invariant measure of critical circle maps with non-integer order and with several break points.

Let $f$ be an orientation preserving homeomorphism of the circle $S^1 = \mathbb{R}^1 / \mathbb{Z}^1$ with lift $F : \mathbb{R}^1 \to \mathbb{R}^1$, which is continuous, strictly increasing and fulfills $F(x + 1) = x + 1$, $x \in \mathbb{R}^1$. The most important arithmetic characteristic of the homeomorphism $f$ of the unit circle $S^1$ is the rotation number:
$$\rho_f = \lim_{n \to \infty} \frac{F^n(x)}{n} \pmod{1}, \quad x \in \mathbb{R}^1.$$

Henceforth, $F^n$ denotes the $n$th iterate of the function $F$. The rotation number is rational if and only if $f$ has periodic orbits. Denjoy proved that if $f$ is a circle diffeomorphism with irrational rotation number $\rho = \rho_f$ and $\log f'$ is of bounded variation, then $f$ is topologically conjugate to the pure rotation $f_\rho : x \to x + \rho \pmod{1}$; that is, there exists an essentially unique homeomorphism $\varphi$ of the circle with $\varphi \circ f = f_\rho \circ \varphi$ (see [1]). Since the conjugating map $\varphi$ and the unique $f$-invariant measure $\mu_f$ are related by $\varphi(x) = \mu_f([0;x])$, $x \in S^1$ (see [1]), regularity properties of the conjugating map $\varphi$ imply corresponding properties of the density of the absolutely continuous invariant measure $\mu_f$ as a distribution function on the circle. The problem of relating the smoothness of $\varphi$ to that of $f$ has been studied extensively. In-
depth results have been found; see [2–5].

Other classes of circle homeomorphisms are critical circle homeomorphisms and circle diffeomorphisms with several break points.

I. Critical Circle Homeomorphisms. The orientation preserving circle homeomorphisms \( f \), such that \( f \in C^r, r \geq 3 \), have a critical point \( x_{cr} \), around which, in some \( C^r \) coordinate system, \( f \) has the form

\[
 f(x) = \phi(x) \mid \phi(x) \mid^{d-1} + f(x_{cr})
\]

for all \( x \in U_{\delta}(x_{cr}) \),

where \( \phi : U_{\delta}(x_{cr}) \rightarrow \phi(U_{\delta}(x_{cr})) \) is a \( C^r \) diffeomorphism such that \( \phi(x_{cr}) = 0 \), and \( d > 1 \).

Such critical point is called non-flat critical point of order \( d \).

An important one-parameter family of examples of critical circle maps are the Arnold’s maps defined by

\[
 f_\theta(x) = x + \theta + \frac{1}{2\pi} \sin 2\pi x \mod 1, \quad x \in S^1
\]

For every \( \theta \in \mathbb{R}^1 \) the map \( f_\theta \) is a critical map with critical point 0 of cubic type.

II. \( P \)-Homeomorphisms. That is, orientation preserving circle homeomorphisms \( f \) are differentiable except in many countable points called break points admitting left and right derivatives (denoted by \( f^- \) and \( f^+ \), resp.) such that

(i) there exist some constants \( 0 < a < b < \infty \) such that

\[
 a < f^-(x) < b \quad \text{for all} \quad x \in S^1 \setminus BP(f) \quad \text{and}
\]

\[
 a < f^+(x) < b \quad \text{for all} \quad x \in BP(f),
\]

where \( BP(f) \) denotes the set of the break points of \( f \);

(ii) \( \log f^+ \) has bounded variation:

\[
 \nu = \text{var log } f^+ < \infty.
\]

The ratio \( \sigma_f(x_b) = \frac{f''(x_b)}{f'_+(x_b)} \) is called jump ratio of \( f \) at \( x_b \).

The existence of the conjugating map for the class critical circle homeomorphisms was proved by Yoccoz in [7] and for the class \( P \)-homeomorphisms the existence of conjugating map was proved by Herman in [2].

The singularity of the conjugating map for critical circle homeomorphisms was shown by Graczyk and Swiątek in [8]. They proved that if \( f \) is \( C^3 \) smooth circle homeomorphism with infinitely many critical points of polynomial type and an irrational rotation number of bounded type, then the conjugating map \( \phi \) is a singular function. For the \( P \)-homeomorphisms, the situation is different; that is, in this case, the conjugating map can be singular or absolutely continuous. Indeed, in the works [9–11], it was shown that the conjugating map is singular. The deeper result in this area was obtained by Dzhalilov et al. [12]. They proved that if \( f \) is piecewise-smooth \( P \)-homeomorphism with infinite number of break points and the product of jump ratios at these break points is nontrivial, then the conjugating map is a singular function. But in the works [9, 13], it was shown that if \( f \) is piecewise-smooth \( P \)-homeomorphism with infinite number of break points having the (D)-property (see for the definition [13]) and the product of the jump ratios on each orbit is equal to 1, then the conjugating map is an absolutely continuous function. Now, we discuss the symmetric property of a given function.

Definition 1. A homeomorphism \( f \) is called quasi-symmetric if there exists a constant \( K > 0 \) such that for any \( x \in S^1 \) and \( \delta \neq 0 \) the following inequality holds:

\[
 \frac{|f(x + \delta) - f(x)|}{|f(x) - f(x - \delta)|} < K.
\]

The criteria of quasi-symmetry of the conjugating map of the critical circle homeomorphisms were obtained by Swiątek in [14]. Due to [14], if the circle homeomorphism with an irrational rotation number is analytic and has infinitely many critical points, then the conjugating map is quasi-symmetric if and only if the rotation number is of bounded type.

The quasi-symmetric property of the conjugating map of
$P$-homeomorphisms is also different from the case of critical circle homeomorphisms. More precisely, if the rotation number of $P$-homeomorphism is irrational of bounded type, then conjugating map is quasi-symmetric, but there is a $P$-homeomorphism with irrational rotation number of unbounded type such that the conjugating map is quasi-symmetric. Now, we introduce our class.

Let $f$ be a circle homeomorphism.

(a) $f$ has one critical point polynomial type of order $d > 2$ and

$$f(x) = (x - x_{cr}) \big| x - x_{cr} \big|^{d-1} + f(x_{cr})$$

for some $\delta$-neighborhood $U_{\delta}(x_{cr})$;

(b) $f$ is a $P$-homeomorphism on the set $S^1 \setminus U_{\delta}(x_{cr})$.

Now, we state our main results.

Theorem 1. Suppose that a circle homeomorphism $f$ satisfies the conditions (a)–(b) and the rotation number $\rho_f$ is irrational. Then, there exists circle homeomorphism $\phi : S^1 \to S^1$, such that the functional equation

$$\phi(x + \rho_f) = f(\phi(x)), \ x \in S^1$$

is satisfied. Moreover, $\phi$ is quasi-symmetric if and only if $\rho_f$ is of bounded type.

Note that the result of Theorem 1 was obtained by Dzhalilov, Noorani and Akhatkulov [15] for critical circle homeomorphisms with odd order of critical point. In our case the order of critical point can be any real number bigger than 2. The present paper is a continuation of [15] and in a certain sense complements the results obtained in that paper.

References

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